ETHNOMATHEMATICS AND EDUCATION IN AFRICA
Paulus Gerdes

Second edition:

ISTEG
Belo Horizonte
Boane
Mozambique
2014

**Institutionen för Internationell Pedagogik**
(Institute of International Education)

**Stockholms Universitet**
(University of Stockholm)

**Report 97**


**Instituto Superior de Tecnologias e Gestão (ISTEG)**
(Higher Institute for Technology and Management)
Av. de Namaacha 188, Belo Horizonte, Boane, Mozambique

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Photograph on the front cover:

Detail of a Tonga basket acquired, in January 2014, by the author in Inhambane, Mozambique
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Institute of International Education

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ETHNOMATHEMATICS AND EDUCATION
IN AFRICA

Paulus Gerdes

January, 1995

STOCKHOLMS UNIVERSITET

Cover of the original edition by the University of Stockholm
PREFACE

The book *Ethnomathematics and Education in Africa* was originally concluded in 1992 and published in January 1995 by the Institute of International Education (IIE) of the University of Stockholm (Sweden) within the framework of the institutional cooperation that had started in 1991 between the IIE and the Higher Pedagogical Institute / Pedagogical University (Instituto Superior Pedagógico / Universidade Pedagógica) in Mozambique. The book contained a collection of papers presented at conferences in Brazil, Kenya, Lesotho, Mozambique, Norway, and Spain and/or published in international journals such as *For the Learning of Mathematics, Educational Studies in Mathematics* and *Historia Mathematica*. The papers date from the period 1986-1992.

The book has been out of print for many years. Many colleagues contacted me to know if the book was still available.

The realisation of the 5th International Conference on Ethnomathematics in Mozambique in July 2014 constitutes another motif for re-editing the book, that dates from the years of the emergence – in various continents – of ethnomathematics as a research field.

The new edition of the book *Ethnomathematics and Education in Africa* reproduces the eleven chapters of the original edition and contains two chapters not included in the first edition. The new chapters are entitled “Conditions and strategies for emancipatory mathematics education in underdeveloped countries” and “African slave and calculating prodigy: Bicentenary of the death of Thomas Fuller.”

The second paper, on Thomas Fuller (1710-1790), was written together with the late John Fauvel (1947-2001), and published in *Historia Mathematica*, New York, Vol. 17, 141-151 (1990).

The new edition of *Ethnomathematics and Education in Africa* will be available both in printed form and as an eBook.

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ISTEG, Boane, Mozambique
January 2014

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Chapter 1

INTRODUCTION

The Secretary of the International Study Group on Ethnomathematics, Patrick Scott (University of New Mexico, USA) explained during the hearing ‘What can we expect from Ethnomathematics?’ (6th International Congress on Mathematics Education, Budapest, 1988) that there are “two major models in ethnomathematics: the D’Ambrosio–Gerdes model (ethnomathematics in Third World Countries to bridge the gap between the home culture and ‘modern’ science) and the Zaslavsky–Ascher model (ethnomathematics in First World countries to bring the world into the classroom and to have an appreciation of the mathematics of other cultures).”¹ Both models may be considered as complementary: both industrialised and Third World countries see themselves nowadays confronted with the need to ‘multi-culturalise’ their mathematics curricula.

In general, ethnomathematics is the study of the interrelationship between (the) mathematics and (the) culture(s) of a given people or population group. Ethnomathematical studies in Third World countries and in Africa, in particular, look for and analyse

* Mathematical traditions that survived colonization and mathematical activities in people’s daily life and ways to incorporate them into the curriculum;
* Culture elements that may serve as a starting point for

doing and elaborating mathematics in the classroom.

The ultimate goals of these studies are to improve the quality of mathematics education, to make it more interesting, to enhance the (cultural) confidence of the pupils in their individual and collective mathematical abilities, to improve the pupils’ motivation, and to accelerate the access of the people of Mozambique (and other Third World countries) to ‘World Mathematics’, the common heritage of mankind, as a (possibly) useful instrument to improve the quality of life.

In Mozambique, ethnomathematical research has taken place since the end of the 1970s. In 1988 it became organized as the ‘Ethnomathematics in Mozambique’ research project that envisages the following general objectives:

1. To reconstruct and to analyse mathematics in African cultures in general and in Mozambican cultures in particular;
2. To contribute to the elaboration of the methodology for this reconstruction;
3. To analyse and to experiment (with) the possibilities of ‘embedding’ or ‘incorporating’ African cultural elements into mathematical education;
4. On the basis of the reconstructed mathematics, to reflect on:
   4.1 didactical alternatives;
   4.2 the (early) history of mathematics;
   4.3 philosophical problems of mathematics;
   4.4 the involved mathematics.

Most ‘mathematical’ traditions that survived colonization and most ‘mathematical’ activities in the daily life of the Mozambican people are not explicitly mathematical. The mathematics is ‘hidden’. The first aim of the project is to ‘uncover’ some of this ‘hidden’

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2 Supported by SAREC (Sweden)
mathematics. As many traditions are nowadays rather obsolete, the ‘uncovering’ often also meant a tentative reconstruction of past knowledge.

In my Ph.D thesis *On the awakening of geometrical thinking* (1985) and in my books *Ethnogeometrie* and *Cultura e o despertar do pensamento geométrico* (Culture and the awakening of geometrical thinking) some anthropological research methods were developed in order to ‘uncover’ and reconstruct ‘hidden’ mathematical thinking.

In a series of papers presented at regional and international conferences I described my experiences with the cultural embedment of mathematical education in an African context. Nine of these educational papers have been included in this collection *Ethnomathematics and Education in Africa*.

In chapter 2 ‘Ethnomathematical research: preparing a response to a major challenge to mathematics education in Africa’ the societal and cultural background of mathematical education in Africa is summarized in order to explain the urgency of ethnomathematical research. It presents also a short review of the realized and ongoing investigations in Mozambique. Chapter 3 ‘On the concept of Ethnomathematics’ discusses different notions of ethnoscience and ethnomathematics. With chapter 4 ‘How to recognize hidden geometrical thinking: a contribution to the development of an anthropology of mathematics’ the questions of who sets the standards in mathematics and mathematics education and of who decides what mathematics is, are posed, and it reflects about the implications for education in an African context. In chapter 5 ‘On culture, geometrical thinking and mathematics education’ a summary of our experimentation with the incorporation of traditional African cultural

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3 Franzbecker Verlag, Bad Salzdethfurth, Germany, 1992, 360 pp.
elements into mathematics education is presented. The paper confronts a widespread prejudice about mathematical knowledge, that mathematics is ‘culture-free’, by demonstrating alternative constructions of Euclidean geometrical ideas developed from the traditional culture of Mozambique. As well as establishing the educational power of these constructions, the paper illustrates the methodology of ‘cultural conscientialization’ in the context of teacher training. In chapter 6 ‘A widespread decorative motif and the Pythagorean Theorem’ concrete examples are given of multiculturalising the mathematics curriculum, using a well-known African and also Scandinavian ornamental motif as a starting point for doing and elaborating mathematics in the classroom. This line is continued in chapter 7 ‘Pythagoras, similar triangles and the “elephants’-defence”-pattern of the (Ba)Kuba’ where it is shown how ornaments from the (Ba)Kuba (Congo) may be used to re-invent the famous Pythagorean Theorem in an educational context. In chapter 8 ‘On possible uses of traditional Angolan sand drawings in the mathematics classroom’ some examples of using the Tchokwe (Cokwe) sand drawings (‘sona’) in the mathematics classroom are suggested. They range from the study of arithmetical relationships, symmetry, similarity, and Euler graphs to the determination of the greatest common divisor of two natural numbers. In chapter 9 ‘Exploration of the mathematical potential of SONA’ an elaborated example of stimulating cultural awareness in mathematics teacher education is presented. In chapter 10 an example of the relationship between mathematics education and technology, art and games is presented. In chapter 11 ‘On the history of mathematics in Africa south of the Sahara’ I give an overview of mathematics and of other African traditions with mathematical ingredients like numeration systems, riddles, puzzles, art and symmetries, games, architecture, ‘sand drawings’, string figures, etc., that may be incorporated into or be used as a starting point for doing, inventing and elaborating mathematical ideas and methods within and outside the formal school context.

At this point my thanks go to Vinayagum Chinapah and Holger Daun for their invitation to publish a volume with the aforementioned papers as an edition of the University of Stockholm’s Institute of International Education, under the title Ethnomathematics and Education in Africa.
I am grateful to SAREC (Swedish Agency for Research Cooperation with Developing Countries), and in particular to Berit Olsson, for the financial support to the ‘Ethnomathematics in Mozambique’ and ‘Archive for the History of Mathematics in Africa’ research projects.

I am indebted to many colleagues all over the world for their critiques and encouragement.

To the many generations of nameless African artisans, artists and scientists who inspired my research and experimentation, I wish to dedicate this volume.

I record my gratitude to my family, in particular to my wife Marcela and to my daughter Lesira for the inspiration they gave me and for the patient support and tolerance they have shown.

Paulus Gerdes
Higher Pedagogical Institute / Instituto Superior Pedagógico
Maputo, Mozambique, September 1992
Chapter 2

ETHNOMATHEMATICAL RESEARCH:
PREPARING A RESPONSE TO A MAJOR
CHALLENGE TO MATHEMATICS EDUCATION IN
AFRICA *

Societal and educational background

Three important publications on the challenges to the South in general and to education in Africa in particular appeared in 1990:

* "The Challenge to the South", The Report of the South Commission, led by the former President of Tanzania, Julius Nyerere 1;

* "African Thoughts on the Prospects of Education for All", selection from papers commissioned for the Regional Consultation on Education for All, Dakar, 27-30 November 1989 2;

* "Educate or Perish: Africa’s Impasse and Prospects”, study directed by Joseph Ki-Zerbo. 3


These profound studies delineate the societal and educational background that certainly has to be taken into account in any reflection on “Mathematics Education in Africa for the 21st Century”.

“The challenge to the South” criticizes development strategies that minimize cultural factors. Such strategies only provoke indifference, alienation and social discord. The development strategies followed until today “have often failed to utilize the enormous reserves of traditional wisdom and of creativity and enterprise in the countries of the Third World”. Instead, the cultural wellsprings of the South should feed the process of development (p. 46).

An important feature of “African thoughts...” is the fact that two themes keep recurring in all contributions: the focus on the crisis in African contemporary culture and the theme of African languages (as vehicles of culture and media of education). The crux of the crisis of African cultures is the issue of African cultural identity (p. 9). A people’s cultural identity (including their awareness of such an identity), is seen as the springboard of their development effort (p.10). Africa needs culture-oriented education that would ensure the survival of African cultures, if it emphasized originality of thought and encouraged the virtue of creativity (p.15). Scientific appreciation of African cultural elements and experience is considered to be “one sure way of getting Africans to see science as a means of understanding their cultures and as a tool to serve and advance their cultures” (p.23).

“Educate or Perish ...” shows that today’s African educational system favours foreign consumption without generating a culture that is both compatible with the original civilization and truly promising. Unadapted and elitist, the existing educational system feeds the crisis by producing economically and socially unadapted people, and by being heedless of entire sections of the active population. Education for all, as discussed by Ki-Zerbo, should be an attempt to encourage the development of initiative, curiosity, critical awareness, individual responsibility, respect for collective rules, and a taste for manual work. Africa needs a “new educational system, properly rooted in both society and environment, and therefore apt to generate the self-confidence from which imagination springs” (p.104).

Reminding us of the apt African proverb “When lost, it’s better to return to a familiar point before rushing on”, Ki-Zerbo underlines
that “Africa is in serious trouble, not because its people have no foundations to stand on, but because ever since the colonial period, they have had their foundations removed from under them” (p. 82). Probably this is particularly true in the case of mathematics. Here lies one of the principal challenges to African mathematics educators.

_A major challenge to mathematics education_

African countries face the problem of low ‘levels’ of attainment in mathematics education. Math anxiety is widespread. Many children (and teachers too?) experience mathematics as a rather strange and useless subject, imported from outside Africa.

The African mathematical heritage, traditions and practices have to be ‘embedded’ or ‘incorporated’ into the curriculum. More

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5 Cf. e.g.:


generally, both in the North and in the South, it is becoming more and more understood that it is necessary to multi-culturalise the mathematics curriculum in order to improve its quality, to augment the cultural and social self-confidence of all pupils (cf. e.g. D’Ambrosio, Bishop, Mellin-Olsen, Zaslavsky). How to respond to this challenge?

* Shirley, L. (1988): Historical and ethnomathematical algorithms for classroom use, Ahmadu Bello University, Zaria (Nigeria), 12 pp. (mimeo);

Cf. e.g.:
* D’Ambrosio, U. (1985): Socio-cultural bases for mathematics education, University of Campinas, Campinas (Brazil), 103 pp.;
Ki-Zerbo stresses (p. 87) that all educational renovation in Africa has to be based on research. This appeal is indeed necessary, as, according to Hagan in “African thoughts.”, “In Africa, there is generally a surprising lack of research to back up proposals for educational reforms” (p. 24). It is in the context of trying to respond to this challenge to mathematics education in Africa, that ethnomathematical research started in Mozambique.  


On the development of mathematics education in Mozambique, see e.g.
Ethnomathematical studies analyse * Ethnomathematical studies analyse * Ethnomathematical studies analyse * Ethnomathematical studies analyse * Ethnomathematical studies analyse * Ethnomathematical studies analyse * Ethnomathematical studies analyse

* Mathematical traditions that survived colonization and mathematical activities in people’s daily life and ways to incorporate them into the curriculum;

* Cultural elements that may serve as a starting point for doing and elaborating mathematics in and outside school.

As most ‘mathematical’ traditions that survived colonization and most ‘mathematical’ activities in the daily life of the Mozambican people are not explicitly mathematical (i.e. the mathematics is ‘hidden’), the first aim of our Ethnomathematics Research Project is to ‘uncover’ this ‘hidden’ mathematics. The first results of this ‘uncovering’ are included in book On the awakening of geometrical thinking (1985) ⁹ and slightly extended in Ethnogeometry: Cultural-

* Draisima, J. et al. (1986): Mathematics Education in Mozambique, Proceedings SAMSA 4, Kwaluseni (Swaziland), 56-96;


⁹ Zum erwachenden geometrischen Denken, Maputo / Dresden, 260 pp. Cf. also Gerdes, P.: How to recognize hidden geometrical thinking? A contribution to the development of anthropological mathematics, For the Learning of Mathematics, Montreal, 1985, Vol. 6, No. 2, 10-12, 17 [reproduced as chapter 4]. Cf. also the following two papers where we analyse why
anthropological contributions to the genesis and didactics of geometry (1992).  


In: *For the Learning of Mathematics*, Montreal, Vol. 8, No. 1, 35-39 [reproduced as chapter 6].
Pythagorean Proposition do there exist?” \(^{14}\) and more elaborated in the book *African Pythagoras. A study in culture and mathematics education: cultural starting points* (1992) \(^{15}\) we show how diverse African ornaments and artefacts may be used to create a rich context for the discovery and the demonstration of the so-called Pythagorean Theorem and of related ideas and propositions.

In *SONA Geometry: Reflections on a Drawing Tradition in Africa south of the Equator* (1993) \(^{16}\) it is tried to reconstruct mathematical components of the Tchokwe drawing-tradition (Angola) and to explore their educational, artistic and scientific potential. In an earlier article “On possible uses of traditional Angolan sand drawings in the mathematics classroom” (1988) \(^{17}\) we analysed already some possibilities for educational incorporation of this tradition. In the paper “Find the missing figures” (1988) \(^{18}\) and in the book *Lusona:*

\(^{14}\) *Namnären,* Göteborg (Sweden), Vol. 15, No. 1, 38-41.

\(^{15}\) Mozambique’s Higher Pedagogical Institute, 1992, 102 pp. [English language edition in colour: Lulu, Morrisville NC, 2011].


geometrical recreations of Africa (1991) mathematical amusements are presented that are inspired by the geometry of the sand drawing tradition. For children (age 10-15) the booklet Living mathematics: Drawings from Africa (1990) has been elaborated. A overview of this research is given in “On mathematical elements in the Tchokwe ‘sona’ tradition” (1990).

In recent years more lecturers and in particular young lecturers, who returned home after having studied abroad became interested in and started ethnomathematical research. At the 3rd Pan-African Congress of Mathematicians (Nairobi, 1991) Abdulcarimo Ismael presented a communication on “The origin of the concepts of ‘even’ and ‘odd’ in Macua culture (Northern Mozambique)” and Marcos Cherinda delivered a paper on “Mental Arithmetic and the Tsonga language (Southern Mozambique)”. At the 8th Symposium of the Southern Africa Mathematical Sciences association (Maputo, 1991) Daniel Soares presented a communication on “Popular counting practices in Mozambique”. At the 1st Conference on Mathematics Education in Africa (Cairo, 1992) Marcos Cherinda presented a paper on “A children’s ‘circle of interest in ethnomathematics” and Jan

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Draisma a communication on “Mental addition and subtraction in Mozambique”.

Also a series of students at Mozambique’s Higher Pedagogical Institute (Maputo and Beira) became interested in ethnomathematical research. Two students completed already a master’s thesis in the field of ethnomathematics: “Symmetries of ornaments on baskets of the ‘khuama’ type” (Evaristo Uaila) and “Symmetries of ornaments on metallic window gratings in the city of Maputo” (Abílio Mapapá).

Ethnomathematical research training will be included in both M.Ed. programmes in Mathematics Education: Mathematics Education for Primary Schools (Beira) and Mathematics Education for Secondary Schools (Maputo). With the expansion of this research it is hoped to contribute to the preparation of a curriculum reform that guarantees that mathematics education in the 21st century in Mozambique is indeed “in tune with African traditions and socio-cultural environment”.

Doctoral theses:
Abdulcarimo Ismael (2001): An ethnomathematical study of Tchadji – about a Mancala type board game played in Mozambique and possibilities for its use in Mathematics Education, University of the Witwatersrand, Johannesburg, South Africa;
Marcos Cherinda (2002): The use of a cultural activity in the teaching and learning of mathematics: The exploration of twill weaving in Mozambican classrooms, University of the Witwatersrand, Johannesburg, South Africa;
Jan Draisma (2006): Teaching gesture and oral computation in Mozambique: four case studies, Monash University, Clayton, Australia;
Daniel Soares (2010): The incorporation of geometry involved in the traditional house building in mathematics education in Mozambique: The cases of the Zambézia and Sofala provinces, University of the Western Cape, South Africa.

“African Thoughts...”, p. 14
Chapter 3

ON THE CONCEPT OF ETHNOMATHEMATICS *

The concept of ethnomathematics is relatively new among mathematicians and teachers of mathematics. The Brazilian U. D’Ambrosio is often called the ‘father of ethnomathematics’ [Cf. e.g. Ferreira, 1988, 3; Borba, 1988, 24]. In many lectures and conferences since 1975 he has stressed the necessity of having an ethnomathematics. On the other hand ethnographers since the end of the last century have used the more general concept of ethnoscience and related notions, like ethnolinguistics, ethnobotany, ethnozoology, ethnochemistry, ethnoastronomy, ethnopsychology and ethno-logics. The usual meaning given by specialists of the social sciences to ethnoscience does, however, generally not correspond to the dominant interpretation of the concept of ethnomathematics by mathematicians, as will be shown in the following.

Ethnographers on ethnoscience

In the ethnological dictionary of Pano and Perrin are presented two definitions of the concept of ethnoscience. In the first case, it is a “branch of ethnology that dedicates itself to the comparison between

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the positive knowledge of exotic societies and the knowledge that has been formalized in the established disciplines of western science” [Panoff & Perrin, 1973, 68]. This definition raises immediately some questions, like: ‘What is positive knowledge?’, ‘In what sense exotic?’, ‘Does there exist a western science?’ In the second case, “each application of one of the western scientific disciplines to natural phenomena which are understood in a different way by indigenous thinking” is called ethnoscience [Panoff & Perrin, 1973, 68]. Both definitions belong to a tradition that traces back to the colonial time when ethnography was born in the most ‘developed’ countries as a ‘colonial science’, that studied almost exclusively the cultures of subjected peoples, also a ‘science’ that opposed the so-called ‘primitive’ thinking to the ‘western’ thinking as somehow absolutely different. M. and R. Ascher, mathematician and ethnographer, did not succeed in liberating themselves completely from this tradition, as they define ethnomathematics as “the study of mathematical ideas of nonliterate peoples”. “We recognize as mathematical thought those notions that in some way correspond to that label in our culture” [Ascher & Ascher, 1986, 125; cf. Ascher, M., 1984]. ‘Our culture’ means ‘western culture’, and they state furthermore that the ‘western’ mathematics did not pass through a nonliterate pre-history: “Ethnomathematics is not a part of the history of Western mathematics” [Ascher & Ascher, 1986, 139]. “The category of mathematics is – in their opinion – ours” (= western [Ascher & Ascher, 1986, 132]. We may question, however, if the ‘oriental’ and ‘eastern’ and ‘southern’ peoples of Asia and Africa, did not contribute to the development of that mathematics considered ‘western.’

Among ethnographers there exists also another current that

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2 Cf. the definition of Lévi-Strauss of ethnozoology: “... the positive knowledge which the natives (of this part of the world) possess concerning animals, the technical and ritual uses to which they put them, and the beliefs they hold about them” [Lévi-Strauss, 1962, 133].

3 Cf. Bishop, 1989, 3: “In a sense that term [western mathematics, pg] is ... inappropriate since many cultures have contributed to this knowledge...”
considers ethnoscience in a very different way. E.g., C. Favrod characterizes ethnolinguistics in his introduction to social and cultural anthropology as follows: “Ethnolinguistics tries to study language in its relationship to the whole of cultural and social life” [Favrod, 1977, 90]. When we transfer this characterization of ethnolinguistics to ethnomathematics, we obtain by analogy: “Ethnomathematics tries to study mathematics (or mathematical ideas) in its (their) relationship to the whole of cultural and social life.” In this sense ethnomathematics comes near to the sociology of mathematics of D. Struik [See Struik, 1986].

**Genesis of the concept of ethnomathematics among mathematicians and mathematics teachers**

Colonial education presented mathematics generally as something rather ‘western’ or ‘European’, as an exclusive creation of ‘white men’. With the hasty curriculum transplantation – during the 1960’s – from the highly industrialized nations to ‘Third World’ countries there continued, at least implicitly, the negation of African, Asian, American-Indian, ... mathematics [Cf. Gerdes, 1985b, #0].

During the 1970’s and 1980’s there emerged among teachers and mathematics educators in developing countries and later also in other countries a growing resistance to this negation [Cf. Njock, 1985],

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4 See, e.g., Harris, 1987, 26: “‘Ex Africa semper aliquid novi’ Pliny is supposed to have written: ‘There is always something new from Africa.’ Part of the newness of Paulus Gerdes’ work in Mozambique [Gerdes 1986a] is that he offers ‘non-standard problems’, easily solved by many illiterate Mozambican artisans, to members of the international mathematics education community, who cannot (at first) do them. They have trouble in constructing angles of 90°, 60° and 45° and regular hexagons out of strips of paper, problems which are no trouble at all to people for whom the intellectual and practical art of weaving is a necessary part of life. Recently I have been offering to
against the racist and (neo)colonial prejudices, that it reflects, against the Eurocentrism in respect to mathematics and its history [Cf. the studies of Joseph, 1984, 1986, 1987a, 1987b, 1991]. It was stressed that beyond the ‘imported school mathematics’ there existed and continues to exist ‘indigenous mathematics’ [Cf. e.g. Gay & Cole, 1967]. In this context various concepts have been proposed to contrast with the ‘academic mathematics’ / ‘school mathematics’ (i.e. the school mathematics of the transplanted, imported curriculum):

* **Sociomathematics** of Africa [Zaslavsky, 1973]: “the applications of mathematics in the lives of African people, and, conversely, the influence that African institutions had upon the evolution of their mathematics” [Zaslavsky, 1973, 7];

* **Spontaneous mathematics** [D’Ambrosio, 1982]: each human being and each cultural group develops spontaneously certain experienced teachers and teachers in training some of the ‘non-standard problems’ that are easily solved by any woman brought up to make her or her family’s clothes. Many of the male teachers are so unfamiliar with the construction and even shape and size of their own garments that they cannot at first perceive that all you need to make a sweater (apart from the technology and tools) is an understanding of ratio and all you need to make a shirt is an understanding of right-angled and parallel lines, the idea of area, some symmetry, some optimisation and the ability to work from 2-dimensional plans to 3-dimensional forms. What makes the problems non-standard is the viewpoint of those who set the standards. Gerdes’ work, and the work of others in the field of ethnomathematics offer a rather threatening confrontation to the traditional standard setters. Gerdes is up against a number of factors that until recently have tried to determine the education, or previous lack of it, in his country. The freshness of his work is his illustration of the mathematics that already exists in Mozambican culture and how he is setting about ‘defrosting’ it. It is interesting to take Gerdes’ analysis and his energy and commitment and to apply them worldwide and in the different context of women’s culture ... ”
mathematical methods 5;  
* Informal mathematics [Posner, 1982]: mathematics that is transmitted and one learns outside the formal system of education;  
* Oral mathematics [Carraher et al., 1982; Kane, 1987]: in all human societies there exists mathematical knowledge that is transmitted orally from one generation to the next;  
* Oppressed mathematics [Gerdes, 1982]: in class societies (e.g., in the countries of the ‘Third World’ during the colonial occupation) there exist mathematical elements in the daily life of the populations that are not recognized as mathematics by the dominant ideology;  
* Non-standard mathematics [Carraher, 1982; Gerdes, 1985; Harris, 1987]: beyond the dominant standard forms of ‘academic’ and ‘school’ mathematics there develops and developed in the whole world and in each culture mathematical forms that are distinct from the established patterns;  
* Hidden or frozen mathematics [Gerdes, 1982, 1985]: although, probably, the majority of mathematical knowledge of the formerly colonized peoples has been lost, one may try to reconstruct or ‘unfreeze’ the mathematical thinking, that is ‘hidden’ or ‘frozen’ in old techniques, like, e.g., that of basket making;  

5 Students and colleagues of D’Ambrosio, like Carraher, Schliemann, Ferreira and Borba published many interesting examples of this spontaneous mathematics.  
6 Cf. the comments of Bishop: “...in many underdeveloped countries and former colonies [there is a response] which is aimed at developing a greater awareness of one’s own culture. Cultural rebirth or cultural ‘conscientisation’ is a recognized goal of the educational process in several countries. Gerdes, in Mozambique, is a teacher educator who has done a great deal of work in this area. He seeks not only to demonstrate interesting mathematical aspects of Mozambican life but also to develop the process of ‘unfreezing’ the ‘frozen’ mathematics, which he uncovers. For example, with
* Folk mathematics [Mellin-Olsen, 1986]: the mathematics (although often not recognized as such) that develops in the working activity of each of the peoples may serve as a starting point in the teaching of mathematics.

The proposals of new concepts are provisional. They belong to a tendency that started in the ‘Third World’ and that later on found an echo in other countries.

The various aspects illuminated by the aforementioned provisional concepts have been gradually united under the more general ‘common denominator’ of ethnomathematics. This process has been accelerated by the creation of the International Studygroup on Ethnomathematics [ISGEm] in 1985. Mathematicians, including the author that tried to avoid the use of the term ethnomathematics because of its connotation with the first ethnographical interpretation of the same concept [see above], saw themselves ‘obliged’ to start to use it more and more. The international debate on what ethnomathematics means became more intense, and during the International Congress of Education Mathematics (Budapest, 1988) various papers on ethnomathematics were presented and also a hearing with the participation of U. D’Ambrosio (Brazil), M. Fasheh (Palestina), P. Gerdes (Mozambique), M. Harris (Great-Britain) and P. Scott (USA), around ‘What can we expect from ethnomathematics?’ [See Bishop et al., 1988] was realized.

**Concept, accent or movement?**

According to the editorial comment, entitled ‘Ethnomathematics: What might it be?’, in the first edition of the newsletter of ISGEm, ethnomathematics lies at the confluence of the plaiting methods used by fishermen to make their fish traps, he demonstrates significant geometric ideas which could easily be assimilated into the mathematics curriculum in order to create what he considers to be a genuine Mozambican mathematics education for the young people there” [Bishop, 1989. 13].
mathematics and of cultural anthropology (ethnography). At a first level, it may be called ‘mathematics-in-the-(cultural)-environment’ or ‘mathematics-in-the-community’ [ISGEm-Newsletter, 1985, Vol. 1, No. 1, 2]. In this way, both sociomathematics, and folk, spontaneous, informal, oral, frozen, non-standard and oppressed mathematics belong to ethnomathematics. At a second level, related to the first one, ethnomathematics is “the particular (and perhaps peculiar) way that specific cultural groups go about the tasks of classifying, ordering, counting and measuring” [ISGEm-Newsletter, 1985, Vol. 1, No. 1, 2].

Some researchers try to unite both levels into one definition. E.g. A. Hunting, in his book ‘Learning, aboriginal world view, and ethnomathematics’ (1985), considers ethnomathematics to be the “Mathematics used by a defined cultural group in the course of dealing with environmental problems and activities” [cited in ISGEm-Newsletter, 1986, Vol. 2, No. 1, 3]. In the opinion of Ferreira, well known for his studies of mathematical activities among Brazilian Indians, ethnomathematics is “mathematics incorporated in popular culture” [Ferreira & Imenes, 1986, 4]. Borba, author of an interesting study about the mathematical knowhow of a ‘favela’ population, defines ethnomathematics as “mathematics practiced by cultural groups, as tribal societies, workgroups or neighbourhoods” [Borba, 1988, 20] and sees ethno-mathematics “as a domain of knowledge intrinsically allied to a cultural group and its interests, that is closely linked to its reality and is expressed in a language that is generally different from the one used by Mathematics seen as a science, in a language that is umbilically bound to its culture, to its ethnicity” [Borba, 1987, 38]. These definitions come very close to one of D’Ambrosio: “...different forms of mathematics which are proper to cultural groups we call Ethnomathematics” [D’Ambrosio, 1987, 5]. According to these authors the following holds for each ethnomathematics:

\[
\text{ethnomathematics} \subset \text{mathematics.}
\]

At the same time, however, the investigation of a concrete ethnomathematics is also called ethnomathematics. In this way writes the
same Ferreira in 1986, that ethnomathematics constitutes a branch of ethnology:

\[\text{ethnomathematics} \subset \text{ethnology},\]

as it analyses “the mathematical knowledge, practised in the daily life of a social group” [Ferreira, 1986, 2]. In this sense, the interpretation of ethnomathematics approximates the definition of ethnomathematics that we deduced above from the definition of ethnolinguistics given by the ethnographer Favrod: “Ethnomathematics tries to study mathematics in its relations with the whole of cultural and social life.” In the same essay, Ferreira considers ethnoscientific also as “a method that may be used to arrive at the concepts of the institutionalized sciences” [Ferreira, 1986, 3]. This implies that ethnomathematics belongs to the didactics of mathematics:

\[\text{ethnomathematics} \subset \text{didactics of mathematics}.\]

The same idea is also stressed at the end of the aforementioned editorial ‘Ethnomathematics: What could it be?’: “... examples of Ethno-mathematics derived from culturally identifiable groups, and related inferences about patterns of reasoning and models of thought can lead to curriculum development projects that build on the intuitive understandings and practiced methods students bring with them to school. Perhaps the most striking need for such curriculum development may be in Third World countries, yet there is mounting evidence that schools in general do not take advantage of their students’ intuitive mathematical and scientific grasp of the world” [ISGEm-Newsletter, 1985, Vol. 1, No. 1, 2].

How may ethnomathematics satisfy simultaneously the ‘conditions’ (1), (2) and (3)? In other words, how may ethnomathematics belong at the same time to mathematics, to ethnology and to the didactics of mathematics?

When A. Bishop compares the definition of Ascher & Ascher (Ethnomathematics as the study of mathematical ideas of nonliterate
peoples [Ascher & Ascher, 1986, 125; cf. Ascher, 1984]) with one of D’Ambrosio (Ethnomathematics as a more local collection of mathematical ideas, that maybe are not (yet) as developed and systematized, as those of the ‘mainstream’ of mathematics [Bishop, 1989, 2,3], he concludes that the concept of ethnomathematics does not yet constitute a well defined term and that “in view of the ideas and data we now have, perhaps it would be better not to use that term but rather to be more precise about which, and whose, mathematics one is referring to in any context” [Bishop, 1989, 13].

When we compare the distinct ‘levels’ (1), (2) and (3), on which one interprets ethnomathematics, we may agree with the appeal of Bishop for caution. Maybe it is better to speak provisionally about an ethnomathematical accent in research and in mathematics education, or of an ethnomathematical movement, that we may characterize as follows:

* ‘Ethnomathematicians’ emphasize and analyse the influences of socio-cultural factors on the teaching, learning and development of mathematics;

* With the notion of ethnomathematics, one draws attention to the fact that mathematics (its techniques and truths) is a cultural product; one stresses that every people – every culture and every subculture – develops its own particular mathematics. Mathematics is considered to be a universal, pan-human activity [Ascher & Ascher, 1981, 159]. As a cultural product mathematics has a history. Under certain economic, social and cultural conditions, it emerged and developed in certain directions; under other conditions, it emerged and developed in other directions. In other words, the development of mathematics is not unilinear [Ascher & Ascher, 1986, 139, 140].

* ‘Ethnomathematicians’ emphasize that the school mathematics of the transplanted, imported ‘curriculum’ is apparently alien to the cultural traditions of Africa, Asia and South America. Apparently this mathematics comes from the outside of the ‘Third World’. In

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7 Cf. the earlier studies of the ethnologist White and of the mathematician Wilder.
reality, however, an important of the contents of this school mathematics is of African and Asian origin. Firstly, it became expropriated in the process of colonization that destroyed greatly the (scientific) culture of the oppressed peoples \(^8\) [Cf. Gerdes, 1985b]. Then colonial ideologies ignored or despised the survivals of African, Asian and American-Indian mathematics. The mathematical capacities of the peoples of the ‘Third World’ became negated or reduced to rote memorization. This tendency has been reinforced by the curriculum transplantation (‘New Math’) from the highly industrialized nations to ‘Third World’ countries in the 1960’s.

* ‘Ethnomathematicians’ try to contribute to the knowledge of the mathematical realizations of the formerly colonized peoples. They look for culture elements, that survived colonialism and that reveal mathematical and other scientific thinking. They try to reconstruct these mathematical thoughts.

* ‘Ethnomathematical studies’ in ‘Third World’ countries look for mathematical traditions that survived colonization and for mathematical activities in people’s daily life and analyse ways to incorporate them into the curriculum.

\(^8\) Cf. e.g. Bishop: “One of the greatest ironies … is that several different cultures and societies contributed to the development of [the so-called] Western Mathematics – the Egyptians, the Chinese, the Indians, the Moslems, the Greeks as well as the Western Europeans. Yet when Western cultural imperialism imposed its version of Mathematics on the colonized societies, it was scarcely recognisable as anything to which these societies might have contributed …” [Bishop, 1989, 14].

\(^9\) Cf. e.g.: Ascher & Ascher, 1981; Bassanezi & Faria, 1988; Marschall, 1987; Closs, 1986; Doumbia, 1988; Njock, 1985; Villadiego, 1984; Gerdes, 1985b, 1986g, 1989 a, b, e.

\(^10\) Cf. e.g.: Borba, 1987, 1988; Carraher, 1982; Ferreira & Imenes, 1986; Ferreira, 1986, 1988a and b; Shirley, 1988; Bishop, 1988a
* Ethnomathematical studies also look for other culture elements that may serve as a starting point for doing and elaborating mathematics in the classroom.

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Chapter 4

HOW TO RECOGNIZE HIDDEN GEOMETRICAL THINKING: A CONTRIBUTION TO THE DEVELOPMENT OF AN ANTHROPOLOGY OF MATHEMATICS *

Confrontation

You are mathematics educators, are you not? So let us see if you are good at mathematics.

* Do you know how to construct a circle given its circumference?

* Do you know how to construct angles that measure $90^\circ$, $60^\circ$ or $45^\circ$, using only the strips of paper I have distributed to you?

* What is the minimum number of strips of paper you need in order to be able to plait a broader strip?

* Can you fold an equilateral triangle out of a square of paper?

* Do you know how to construct a regular hexagon out of paper strips?

I gave you five minutes. Who solved all the problems? Nobody?

How is that possible?

* This chapter is an abridged version of an invited paper, ‘On culture, mathematics and curriculum development in Mozambique’, presented at the International Seminar on Mathematics and Culture, Bergen, Norway (1985); the chapter was published in: *For the Learning of Mathematics*, Montreal, Vol. 6, No. 2, 1986, 10-12, 17.
Who solved four problems? Nobody? Three of them? ... You failed? Do you not have the necessary mathematical abilities? No, that is not the reason; you need more time, don’t you? But you are mathematicians, are you not?

You need more time to analyse these non-standard problems. All right. But let me say to you that many of our (illiterate) Mozambican artisans know how to solve these problems ... (obviously “formulated” in another way).

Introduction

The President of the Interamerican Committee on Mathematical Education, Ubiratan D’Ambrosio, has stressed the need for the recognition, incorporation and compatibilization of ethnomathematics into the curriculum [e.g., D’Ambrosio, 1984, p. 10]. This integration of mathematical traditions

“requires the development of quite difficult anthropological-research methods relating to mathematics” [D’Ambrosio, 1985, p. 47];

Anthropological mathematics...constitutes an essential research theme in Third World countries ... as the underlying ground upon which we can develop curriculum in a relevant way” [D’Ambrosio, 1985, p. 47].

In order to be able to incorporate popular (mathematical) practices into the curriculum, it is first of all necessary to recognize their mathematical character. Traditional counting methods, e.g. by means of knots in strings, and counting systems are relatively easily recognized as mathematics. But what about geometrical thinking?

Traditional Mozambican houses have conical roofs and circular or rectangular bases. Rectangular mats are rolled up into cylinders. Baskets possess circular borders. Fish traps display hexagonal holes. Could these examples figure in the mathematics lesson as illustrations of geometrical notions?

Only as illustrations?
This is a rather fundamental question that has recently also been posed by Howson, Nebres and Wilson in their discussion paper on *School mathematics in the 1990s*:

There has been increasing talk, particularly with respect to developing countries, of “ethnomathematics”, i.e. mathematical activities identified within the everyday life of societies. Thus, for example, a variety of types of symmetries are used for decoration in all cultures, numerous constructions are erected which illustrate mathematical laws. To what extent are these activities really “mathematical”? What is it which makes the activities “mathematical” rather than, say, “capable of mathematical elaboration or legitimisation”? [Howson *et al.*, 1985, p. 15]

In order to answer this question, let us analyse some examples.

**First example**

![Figure 1](image-url)

Figure 1

Take two strips of paper in your hands. How do you have to fold them around another in order to be able to weave them further (see Figures 1 and 2)? What has to be the initial angle between the two strips? Vary the angle. What do you discover?
Only one special angle makes further plaitsing possible (see Figures 3, 4 and 5). Two types of strips can be woven in this way (see Figures 6 and 7). The strip pattern in Figure 7 admits changes in direction, like the “circling around” in Figure 8. It is exactly this possibility that makes this strip weaving process very useful. For example, Mozambican artisans use this method for making their straw hats by knitting together the successive wings of a plaited spiral.
Now I repeat the question. Can this result only be used in the mathematics lesson as an illustration of geometrical notions? What is your answer?

When discovering the strip weaving method, did you do mathematics?
Did you analyse the effects of the variation of the angle between the two initial strips of paper?

Let’s go further. What can be said about that particular necessary angle between the two strips? Observe the resulting strip. That particular angle goes three times into a straight angle (see Figure 9). The little triangles possess three of those angles, and therefore ...

What other geometrical knowledge can be obtained? (see e.g. Figure 10).
Second example

Consider the following practical problem. In many situations it is disadvantageous to have a densely woven basket, e.g. when transporting small birds in a basket, they must be able to breathe. So it is useful to have a basket with holes. A basket with holes will also be less heavy. Can you weave a basket with holes?

Like in Figure 11?

![Figure 11](image1.png)

Are the holes fixed? More or less flexible? This may be permitted? Why not? How may the problem be solved?

![Figure 12](image2.png)
Maybe by weaving in more than two directions? What happens when you introduce supporting strands? E.g. in a diagonal way? (see Figure 12). How do you have to introduce them so that the holes become fixed? Is it possible to adapt the three directions in such a way that they become more “equal”?

The resultant regular hexagonal pattern is exactly the one Mozambican peasants use for their light transportation baskets and fishermen for their fish traps.

Do we do mathematics?

Do you still doubt? Please suspend your judgement for a while longer.

Let us solve together another practical production problem.

**Third example**

How can you fasten a border to the walls of a basket when both border and walls are made out of the same material?

Try to solve this problem for yourself. Take two equal strips of paper in your hands, and consider one of them as part of the border, the other as belonging to the wall. How should one join them together?

Should we join the border- and wall-strips as in Figure 13? No ... It is necessary to wrap the wall-strip once more around the border-strip. In this way (Figure 14)? No? How then? As in Figure 15?
But what happens when you flatten the wall-strip (see Figure 15)? How can we avoid the problem? What has to be the initial angle between the border- and the wall-strip (Figure 16)?

Let us complete the border and the wall. What happens? See Figure 17. Are there any other possibilities? Introducing more horizontal strips ... what now? Once again a hexagonal pattern appears.
What other geometrical knowledge can be obtained? Possibility of a hexagonal tiling pattern (Figure 19), etc.

Did you do mathematics?

Let us try to draw some conclusions from these few examples [many other examples can be given, see Gerdes, 1985b].

A method for recognizing hidden geometrical thinking

In our analysis of the geometrical forms of traditional – Mozambican – objects, like baskets, mats, pots, houses, fish traps, etc., we posed the question: why do these material products possess the forms they have? In order to answer this question, we learned the usual production techniques and tried to vary the forms. It emerged that the
forms of these objects are almost never arbitrary, but generally possess many practical advantages, and are, a lot of the time, the only possible or the optimal solutions of specific production problems, as in the examples we have given. The traditional forms reflect accumulated experience and wisdom. They constitute an expression not only of biological and physical knowledge about the materials that are used, but also of mathematical knowledge. [The first results from this research are summarized in Gerdes, 1985b.]

![Figure 19](image.png)

**Cultural and pedagogical value**

*There exists* “hidden” or “frozen” mathematics. The artisan who imitates a known production technique is – generally – doing some mathematics. The artisans who discovered the technique, *did* quite a lot of mathematics, *developed* mathematics, were thinking mathematically.

By unfreezing this frozen mathematics, by rediscovering hidden mathematics in the Mozambican culture, we show indeed that the people of Mozambique, like every other people, did mathematics. After so many years of colonial repression of the culture one encourages, by defrosting the frozen mathematics, an understanding that the people of Mozambique – and other formerly colonized peoples – were capable of developing mathematics in the past, and therefore –
regaining cultural confidence [cf. Gerdes, 1982, 1985a] – will be capable, now and in the future, of developing and using mathematics creatively.

Defrosting frozen mathematics can serve as a starting point for doing and elaborating mathematics in the classroom, as we showed in the geometrically-related examples we gave.

At the same time “unfreezing frozen mathematics” forces mathematicians and philosophers to reflect on the relationship between geometrical thinking and material production, between doing mathematics and technology. Where do (early) geometrical ideas come from? [cf. Gerdes, 1985b].

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Chapter 5

ON CULTURE, GEOMETRICAL THINKING AND MATHEMATICS EDUCATION * 1

* This article is dedicated to Samora Machel, President of Mozambique, who died on October 19, 1986, the day of conclusion of this article.

“Colonization is the greatest destroyer of culture that humanity has ever known ... long-suppressed manifestations of culture have to regain their place ...” (Samora Machel, 1978).

“Education must give us a Mozambican personality which, without subservience of any kind and steeped in our own realities, will be able, in contact with the outside world, to assimilate critically the ideas and experiences of other peoples, also passing on to them the fruits of our thought and practice” (Samora Machel, 1970).

Some social and cultural aspects of mathematics education in Third World countries

In most formerly colonized countries, post-independence education did not succeed in appeasing the hunger for knowledge of its people’s masses.

Although there had occurred a dramatic explosion in the student population in many African nations over the last twenty five years, the mean illiteracy rate for Africa was still 66% in 1980. Overcrowded classrooms, shortage of qualified teachers and lack of teaching materials, contributed towards low levels of attainment. In the case of mathematics education, this tendency has been reinforced by a hasty *curriculum transplantation* from highly industrialized nations to Third World countries. With the transplantation of curricula their *perspective* was also copied: “(primary) mathematics is seen only as a stepping stone towards secondary mathematics, which in turn is seen as a preparation for university education” (Broomes and Kuperes 1983, p. 709). Mathematics education is therefore structured in the interests of a social élite. To the majority of children, mathematics looks rather useless. Maths anxiety is widespread; especially for sons and daughters of peasants and laborers, mathematics enjoys little popularity. Mathematics education serves the selection of élites: “Mathematics is universally recognized as the most effective education filter,” as El Tom underlines (El Tom, 1984, p. 3). Ubiratan D’Ambrosio, president of the Interamerican Committee on Mathematics Education agrees: “... mathematics has been used as a *barrier to social access*, reinforcing the power structure which prevails in the societies (of the Third World). No other subject in school serves so well this purpose of reinforcement of power structure as does mathematics. And the main tool for this negative aspect of mathematics education is evaluation” (D’Ambrosio 1983, p. 363).

In their study of the mathematics learning difficulties of the Kpelle (Liberia), Gay and Cole concluded, that there do not exist any inherent difficulties: what happened in the classroom, was that the

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2 Cf. e.g. Eshiwani (1979), Nebres (1983) and El Tom (1984).
contents did not make any sense from the point of view of Kpelle-culture; moreover the methods used were primarily based on rote memory and harsh discipline (Gay and Cole 1967, p. 6). Experiments showed that Kpelle illiterate adults performed better than North American adults, when solving problems, like the estimation of number of cups of rice in a container, that belong to their ‘indigenous mathematics’ (Gay and Cole 1967, p. 66). Serious doubts about the effectiveness of school mathematics teaching are also raised by Latin American researchers. Eduardo Luna (Dominican Republic) posed the question if it is possible, that the practical mathematical knowledge that children acquired outside the school is ‘repressed’ and ‘confused’ in the school (Luna 1983, p. 4). Not only possible, but this happens frequently, as shown by Carraher and Schliemann in Brazil: children, who knew before they went to school, how to solve creatively arithmetical problems which they encountered in daily life, e.g. at the marketplace, could, later in the school, not solve the same problem, i.e. not solve them with the methods taught in the arithmetic class (Carraher et al. 1982). D’Ambrosio concludes that “‘learned’ matheracy eliminates the so-called ‘spontaneous’ matheracy” (D’Ambrosio 1984, p. 6; Cf. D’Ambrosio 1985b), i.e. “An individual who manages perfectly well numbers, operations, geometric forms and notions, when facing a completely new and formal approach to the same facts and needs creates a psychological blockade which grows as a barrier between the different modes of numerical and geometrical thought” (D’Ambrosio 1984, p. 6, italics P. G.). What happens in the school, is that “the former, let us say, spontaneous, abilities (are) downgraded, repressed and forgotten, while the learned ones (are not being) assimilated, either as a consequence of a learning blockage, or of an early dropout ...” (D’Ambrosio 1984, p. 8, italics P. G.). For this reason, “the early stages of mathematics education (offer) a very efficient way of instilling the sense of failure, of dependency in the children” (D’Ambrosio 1984, p. 7). How can this psychological blockade be avoided?

How can this “totally inappropriate education, leading to misunderstanding and sociocultural and psychological alienation” (Pinxten 1983, p. 173) be avoided? How can this ‘pushing aside’ and ‘wiping out’ of spontaneous, natural, informal, indigenous, folk, implicit, non-standard and / or hidden (ethno)mathematics be avoided?
Gay and Cole became convinced that it is necessary to investigate first the ‘indigenous mathematics’, in order to be able to build effective bridges from this ‘indigenous mathematics’ to the new mathematics to be introduced in the school: “...the teacher should begin with materials of the indigenous culture, leading the child to use them in a creative way” (Gay and Cole 1967, p. 94), and from there advance to the new school mathematics. The Tanzanian curriculum specialist Mmari stresses, that: “... there are traditional mathematics methods still being used in Tanzania ... A good teacher can utilize this situation to underline the universal truths of the mathematical concepts” (Mmari 1978, p. 313). And how could the good teacher achieve this? Jacobsen answers: “The (African) people that are building the houses are not using mathematics; they’re doing it traditionally ... if we can bring out the scientific structure of why it’s done, then you can teach science that way” (Quoted by Nebres 1984, p. 4). For D’Ambrosio, it becomes necessary “... to generate ways of understanding, and methods for the incorporation and compatibilization of known and current popular practices into the curriculum. In other words, in the case of mathematics, recognition and incorporation of ethnomathematics into the curriculum” (D’Ambrosio 1984. p 10). “... this ... requires the development of quite difficult anthropological research methods relating to mathematics; ... anthropological mathematics ... constitutes an essential research theme in Third World countries ... as the underlying ground upon which we can develop curriculum in a relevant way” (D’Ambrosio 1985a, p. 47).
Towards a cultural-mathematical reaffirmation

D’Ambrosio stressed the need for incorporation of ethnomathematics into the curriculum in order to avoid a psychological blockade. In former colonized countries as Mozambique, there exists also a related cultural blockade to be eliminated. “Colonization – in the words of Samora Machel, first President of Mozambique – is the greatest destroyer of culture that humanity has ever known. African society and its culture were crushed, and when they survived they were co-opted so that they could be more easily emptied of their content. This was done in two distinct ways. One was the utilization of institutions in order to support colonial exploitation ... The other was the ‘folklorizing’ of culture, its reduction to more or less picturesque habits and customs, to impose in their place the values of colonialism”. “Colonial education appears in this context as a process of denying the national character, alienating the Mozambican from his country and his origin and, in exacerbating his dependence on abroad, forcing him to be ashamed of his people and his culture” (Machel 1978, p. 401). In the specific case of mathematics, this science was presented as an exclusively white men’s creation and ability; the mathematical capacities of the colonized peoples were negated or reduced to rote memorization; the African and American-Indian mathematical traditions became ignored or despised.

A cultural rebirth is indispensable, as Samora Machel underlines: “... long-suppressed manifestations of culture (have to) regain their place” (Machel 1978, p. 402). In this cultural rebirth, in this combat of racial and colonial prejudice, a cultural-mathematical-reaffirmation plays a part: it is necessary to encourage an understanding that the peoples of the Third World have been capable of developing mathematics in the past, and therefore regaining cultural confidence (Cf. Gerdes 1982, 1985a) – will be able to assimilate and develop the mathematics we need; mathematics does not come from outside the African, Asian and American-Indian cultures.

We may conclude that the incorporation of mathematical traditions into the curriculum will – probably – contribute not only to the elimination of individual and social psychological blockade, but
also of the related cultural blockade. Now, this raises an important question: which mathematical traditions? In order to be able to incorporate popular (mathematical) practices, it is first of all necessary to recognize their mathematical character. In this sense, D’Ambrosio speaks about the need to broaden our understanding of what mathematics is (D’Ambrosio 1985, p. 45). Ascher and Ascher remark in this connection “Because of the provincial view of the professional mathematicians, most definitions of mathematics exclude or minimize the implicit and informal; ...involvement with concepts of number, spatial configuration, and logic, that is, implicit or explicit mathematics, is panhuman” (Ascher and Ascher 1981, p. 159, italics P. G.; cf. Gerdes 1985b, §2).

Broadening our understanding of what may be mathematics, is necessary, but not sufficient. A related problem is how to reconstruct mathematical traditions, when probably many of them have been – as a consequence of slavery, of colonialism... – wiped out. Few or almost none (as in the case of Mozambique) written sources can be consulted. Maybe for number systems and some aspects of geometrical thinking, oral history may constitute an alternative. What other sources can be used? What methodology?

We developed a complementary methodology that enables one to uncover in traditional, material culture some hidden moments of geometrical thinking. It can be characterized as follows. We looked to the geometrical forms and patterns of traditional objects like baskets, mats, pots, houses, fish traps, etc. and posed the question: why do these material products possess the form they have? In order to answer this question, we learned the usual production techniques and tried to vary the forms. It came out that the form of these objects is almost never arbitrary, but generally represents many practical advantages and is, quite a lot of times, the only possible or optimal solution of a production problem. The traditional form reflects accumulated experience and wisdom. It constitutes not only biological and physical knowledge about the materials that are used, but also mathematical knowledge, knowledge about the properties and relations of circles, angles, rectangles, squares, regular pentagons and hexagons, cones, pyramids, cylinders, etc.

Applying this method, we discovered quite a lot of ‘hidden’ or
The artisan, who imitates a known production technique, is, generally, doing some mathematics. But the artisans, who discovered the techniques, did and invented quite a lot of mathematics, were thinking mathematically. When pupils are stimulated to reinvent such a production technique, they may be encouraged to do and learn mathematics. Hereto they can be stimulated only if the teachers themselves are conscious of hidden mathematics, are convinced of the cultural, educational and scientific value of rediscovering and exploring hidden mathematics, are aware of the potential of ‘unfreezing’ this ‘frozen mathematics’. Now we shall present some of our experiences in this necessary ‘cultural conscientialization’ of future mathematics teachers.

Examples of ‘cultural conscientialization’ of future mathematics teachers

Study of Alternate Axiomatic Constructions of Euclidean Geometry in Teacher Training

Many alternate axiomatic constructions for Euclidean geometry have been devised. In Alexandrov’s construction, Euclid’s famous fifth

4 The first results are summarized in Gerdes (1985b). Cf. Gerdes (1986a,f). By bringing to the surface geometrical thinking that was hidden in very old production techniques, like that of basketry, we succeeded in formulating new hypotheses on how the ancient Egyptians and Mesopotamians could have discovered their formulas for the area of a circle [cf. Gerdes (1985b,c, 1986d)] and for the volume of a truncated pyramid [cf. Gerdes (1985b)]. It proved possible to formulate new hypotheses on how the so-called ‘Theorem of Pythagoras’ could have been discovered [cf. Gerdes (1985b, 1986c, e)].

5 Experimental course developed for secondary schools in the Soviet Union (1981) by a team directed by the academician A. Alexandrov.
postulate is substituted by the ‘rectangle axiom’:

\[
\begin{array}{c}
D & C \\
\downarrow & \downarrow \\
A & B
\end{array} \\
, \text{ then} \\
\begin{array}{c}
D & C \\
\downarrow & \downarrow \\
A & B
\end{array}
\]

i.e., if \(AD = BC\) and \(\alpha\) and \(\beta\) are right angles, then \(AB = DC\) and \(\gamma\) and \(\delta\) are also right angles. In one of the classroom sessions of an introductory geometry course, we posed the following provocative question to our future mathematics teachers – many of whom are sons and daughters of peasants: “Which ‘rectangle axiom’ do the Mozambican peasants use in their daily life?” The students’ first reactions were rather skeptical in the sense of “Oh, they don’t know anything about geometry...”. Counter-questions followed: “Do the peasants use rectangles in their daily life?” “Do they construct rectangles?” Students from different parts of the country were asked to explain to their colleagues how their parents construct e.g. the rectangular bases of their houses. Essentially, two construction techniques are common:

(a) In the first case, one starts by laying down on the floor two long bamboo sticks of equal length.

Then these first two sticks are combined with two other sticks also of equal length, but normally shorter than the first ones.
Now the sticks are moved to form a closure of a quadrilateral.

One further adjusts the figure until the diagonals – measured with a rope – become equally long. Then, where the sticks are now lying on the floor, lines are drawn and the building of the house can start.

(b) In the second case, one starts with two ropes of equal length that are tied together at their midpoints.

A bamboo stick, whose length is equal to that of the desired breadth of the house, is laid down on the floor and at its endpoints pins are hit into the ground. An endpoint of each of the ropes is tied to one of the pins.
Then the ropes are stretched and at the remaining two endpoints of the ropes, new pins are hit into the ground. These four pins determine the four vertices of the house to be built.

“Is it possible to formulate the geometrical knowledge, implicit in these construction techniques, into terms of an axiom?” “Which ‘rectangle axiom’ do they suggest?” Now the students arrive at the following two alternate ‘rectangle axioms’:

(a) i.e. if AD = BC, AB = DC and AC = BD, then $\alpha$, $\beta$, $\gamma$, and $\delta$ are right angles. In other words, an equidiagonal parallelogram is a rectangle.
i.e. if $M = AC \cap BD$ and $AM = BM = CM = DM$, then $\alpha$, $\beta$, $\gamma$, and $\delta$ are right angles, $AD = BC$ and $AB = DC$. In other words, an equi-semidiagonal quadrilateral is a rectangle. “After all, our peasants know something about geometry”, remarks a student. Another, more doubtful: “But these axioms are theorems, aren’t they?”.... This classroom session leads to a more profound understanding by the student of the relationships between experience, the possible choices of axioms, between axioms and theorems at the first stages of alternate axiomatic constructions. It prepares the future teachers for discussions later in their study on which methods of teaching geometry seem to be the most appropriate in our cultural context. It contributes to cultural-mathematical confidence.

An Alternate Construction of Regular Polygons

Artisans in the north of Mozambique weave a funnel in the following way. One starts by making a square mat $ABCD$, but does not finish it: with the strands in one direction (horizontal in our figure), the artisan advances only until the middle.
Then, instead of introducing more horizontal strips, he interweaves the vertical strands on the right (between C and E) with those on the left (between F and D). In this way, the mat does not remain flat, but is transformed into a ‘basket’. The center T goes downwards and becomes the vertex of the funnel. In order to guarantee a stable rim, its edges AB, BC, and AC are rectified with little branches. As a final result, the funnel has the form of a triangular pyramid. So far about this traditional production technique.  

We posed our students the following question: “What can we learn from this production technique?” “The square ABCD has been transformed into a triangular pyramid ABC.T whose base ABC is an equilateral triangle. Maybe a method to construct an equilateral triangle?”...

The implicit geometrical knowledge that it reveals is analysed in Gerdes (1985b).
Some reacted skeptically: “A very clumsy method to do so ...”. Counter-questions: “Avoid overhasty conclusions! What was the objective of the artisan? What is our objective?” “Can we simplify the artisans’ method if we only want to construct an equilateral triangle?” “How to construct such a triangle out of a square of cardboard paper?” An answer to these questions is given in the following diagrams:

folding the diagonals

folding FT

join the triangles DFT and CFT until C and D coincide,

F goes up, T goes down
fix the ‘double triangle’ DFT to the face ATC, e.g. with a paperclip

“Can this method be generalized?” “Starting with a regular octagon, how to transform it into a regular heptagonal pyramid?” “How to fold a regular octagon?”

folding the diagonals and FT
F goes up, T goes down and A7 and A8 approximate until they coincide

“How to transform the regular heptagonal pyramid into a regular hexagonal pyramid?” As $2^n$-gons are easy to fold (by doubling the central diagonals when one starts with a square) and each time that the simplified ‘funnel-method’ is applied, the number of sides of a regular polygon (or of the regular polygonal base of a pyramid) decreases by 1, it can be concluded that all regular polygons can be constructed in this way (For more details, see Gerdes 1986b). Once arrived at this point, it is possible to look back and ask: “Did we learn something from the artisans who weave funnels?” “Is it possible to construct a regular heptagon using only a ruler and a compass?” “Why not?” “And with our method?”

“What are the advantages of our general method in relation to the standard Euclidean ruler and compass constructions?” “What are its disadvantages?” “Which method has to be preferred for our primary schools?” “Why?”
By pulling a little lasso around a square-woven button, it is possible to fasten the top of a basket, as is commonly done in southern parts of Mozambique. The square button, woven out of two strips, hides some remarkable geometrical and physical considerations. By making them explicit, the interest in this old technique is already revived. But much more can be made out of it, as will now be shown.

When one considers the square-woven button from above, one observes the pattern (a) or the pattern obtained (b) after rectifying the slightly curved lines and by making the hidden lines visible:

![Image of patterns](image-url)

In its middle there appears a second square. Which other squares can be observed, when one joins some of these square-woven buttons together? Do there appear other figures with the same area as (the top of) a square-woven button?

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Another ‘culturally integrated’ introduction to the ‘Theorem of Pythagoras’ is presented in Gerdes (1986c, g)
Yes, if you like, you may extend some of the line segments or rub out some others.

What do you observe? Equality in areas?
Hence \( C = A + B \):

i.e. one arrives at the so-called ‘Theorem of Pythagoras’.

The teacher-students rediscover themselves this important theorem and succeed in proving it. One of them remarks: “Had Pythagoras – or somebody else before him – not discovered this theorem, we would have discovered it” ... Exactly! By not only making explicit the geometrical thinking ‘culturally frozen’ in the square-woven buttons, but by exploiting it, by revealing its full potential, one stimulates the development of the abovementioned necessary cultural-mathematical (self-)confidence. “Had Pythagoras
not ... we would have discovered it”. The debate starts. “Could our ancestors have discovered the ‘Theorem of Pythagoras’?” “Did they?”... “Why don’t we know it?” ... “Slavery, colonialism ...” By ‘defrosting frozen mathematical thinking’ one stimulates a reflection on the impact of colonialism, on the historical and political dimensions of mathematics (education).

**From Traditional Fish Traps to Alternate Circular Functions, Football and the Generation of (Semi)regular Polyhedra**

Mozambican peasants weave their light transportation baskets ‘litenga’ and fishermen their traps ‘lema’ with a pattern of regular hexagonal holes. One way to discover this pattern is the following. How can one fasten a border to the walls of a basket, when both border and wall are made out of the same material? How to wrap a wall strip around the border strip?

What happens when one presses (horizontally) the wall strip? What is the best initial angle between the border- and wall strip?
In the case that both strips have the same width, one finds that the optimal initial angle measures $60^\circ$. By joining more wall strips in the same way and then introducing more horizontal strips, one gets the ‘litenga’ pattern of regular hexagonal holes.

By this process of rediscovering the mathematical thinking hidden in these baskets and fish traps – and in other traditional production techniques – the future teachers feel themselves stimulated to reconsider the value of their cultural heritage: in fact, geometrical thinking was not and is not alien to their culture. But more than that. This “unfreezing of culturally frozen mathematics” can serve, in many ways, as a starting point and source of inspiration for doing and elaborating other interesting mathematics. In the concrete case of this hexagonal-weaving-pattern, for example, the following sets of geometrical ideas can be developed.

a. Tiling patterns and the formulation of conjectures.

Regular hexagonal and other related tiling patterns can be discovered by the students.
With the so-found equilateral triangle, many other polygons can be built. By considering these figures, general conjectures can be formulated, e.g.

* the sum of the measures of the internal angles of a \( n \)-gon is equal to \( 3(n-2) \times 60^\circ \);

* areas of similar figures are proportional to the squares of their sides;
Once these general theorems are conjectured, there arises the question of justifying, how to prove them.

\[ 1 + 3 + 5 = 9 \]

* the sum of the first \( n \) odd numbers is \( n^2 \).

\[ 1 + 3 = 4 \]

\[ side=1, \ area=1=1^2; \ side=2, \ area=4=2^2; \ side=3, \ area=9=3^2 \]

**b. An alternate circular function.**

Let us return to the weaving of these ‘litenga’ baskets. What happens when the ‘horizontal’ and ‘standing’ strips are of different width, e.g. 1 (unity of measurement) and \( a \)?

One finds a semi-regular hexagonal pattern. How does the optimal angle \( \alpha \) depend on \( a \)?
\[ \alpha = \text{hex}(a) \]

How does \( a \) vary? Both \( \alpha \) and \( a \) can be measured. One finds:

We have here a culturally integrated way to introduce a circular function. After the study of the ‘normal’ trigonometric functions, their relationships can be easily established, e.g.

\[ a = \text{hex}^{-1}(\alpha) = \frac{1}{2 \cos \alpha}. \]

c. Footballs and polyhedra

The faces and edges of the ‘lema’ fish trap display the regular-hexagonal-hole-pattern. At its vertices the situation is different. The artisans discovered that in order to be able to construct the trap, ‘curving’ the faces at its vertices, it is necessary, e.g. at the vertices A, B and C to reduce the number of strips. At these points, the six strips that ‘circumscribe’ one hexagon, have to be reduced to five. That is why one encounters at these vertices little pentagonal holes.
What can be learnt from this implicit knowledge? What types of baskets can be woven, that display at all their vertices pentagonal holes?

It comes out that the smallest possible ‘basket’, made out of six strips, is similar to the well-known modern football made out of pentagonal and hexagonal pieces of leather.

When one ‘planes’ this ball, one gets a truncated icosahedron, bounded by 20 regular hexagons and 12 regular pentagons. By extending these 20 hexagons, one generates the regular icosahedron. On the other hand, when one extends the 12 pentagons, the regular dodecahedron is produced.

What type of ‘baskets’ can be woven, if one augments their ‘curvature’? Instead of pentagonally woven ‘vertices’, there arise
square-hole-vertices. By ‘planing’ the smallest possible ‘ball’, one gets a truncated octahedron, bounded by 6 squares and 8 regular hexagons. Once again, by extension of its faces, new regular polyhedra are discovered, this time, the *cube* and the regular *octahedron*. When one augments still more the curvature of the ‘ball’, there appear triangular-hole-vertices and by ‘planing’ the ‘ball’, one gets a truncated tetrahedron, bounded by 4 regular hexagons and 4 equilateral triangles. By extension of its hexagonal or triangular faces one obtains a regular *tetrahedron*.

Many interesting questions can be posed to future teachers, e.g.
* Is it possible to ‘weave’ other semi-regular polyhedra? Semi-regular, in what sense?
* Did we generate all regular polyhedra? Why?
* What happens if one, instead of reducing the material at a vertex of the basket, augments it?

**Concluding remarks**

Of the struggle against ‘mathematical underdevelopment’ and the combat of racial and (neo)colonial prejudice, a cultural-mathematical reaffirmation makes a part. A ‘cultural conscientialization’ of future mathematics teachers, e.g. in the way we described, seems indispensable.

Some other conditions and strategies for mathematics education to become *emancipatory* in former colonized and (therefore) underdeveloped countries have been suggested elsewhere (Cf. e.g. Gerdes 1985a, 1986a; D’Ambrosio 1985b and Mellin-Olsen 1986).

**Acknowledgements**

The author is grateful to Dr. A. J. Bishop (Cambridge) for his invitation to write this article and to Dr. W. Humbane (Maputo) for proofreading this paper.
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Chapter 6

A WIDESPREAD DECORATIVE MOTIF AND THE PYTHAGOREAN THEOREM *

In their already classical study of the mathematics learning difficulties of the Kpelle (Liberia), Gay and Cole [1967. p.6] concluded that there do not exist any inherent difficulties. What happened in the classroom was that the contents did not make any sense from the point of view of Kpelle culture...

Subsequent research and analyses reinforced this conclusion and recognised that in view of the “educational failure” of many children from Third World countries and from ethnic minority communities in industrialised countries like Great Britain, France and U.S.A., it is necessary to (multi)culturalise the school curriculum in order to improve the quality of mathematics education [cf. e.g. Bishop, 1988; D’Ambrosio, 1985 a, b; Eshiwani, 1979; Gerdes 1985 a, b, 1981 a, 1988 a, b; Ginsburg & Russell, 1981; Mellin-Olsen, 1986; Nebres, 1983; Njock 1985]. In other words, the mathematics curriculum has to be “embedded” into the cultural environment of the pupils. Not only its ethnomathematics, but also other culture elements, may serve as a starting point for doing and elaborating mathematics in the classroom [cf. D’Ambrosio, 1985 a, b; Gerdes, 1986 b, 1988 a, b]. In this article we explore the mathematical-educational potential of such a cultural element: a widespread decorative motif.

A widespread decorative motif

One of the best-known basketry designs of the Salish Indians of

British Columbia is their so-called *star* pattern [Ferrand, 1900, p. 397, Plate XII. See Figure 1]. The Pomo Indians of California used to name it *deer-back* or *potato forehead* [Barrett, 1908, p. 199].

This decorative motif has a long tradition and can be encountered all over the world. The same star pattern is already found in Ancient Egypt [Wilson, 1986, T. 23, p. 21; see Figure 2]. The detail of a beautifully plaited mat from Angola, shown in Figure 3, is known as the *tortoise* design [Bastin, 1961, p. 116]. One encounters the star pattern also on tiled pavements in Arabic-speaking countries.

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1 Its possible technical origin in the production or flat, circular baskets is analysed in Gerdes [1985, pp. 47-51, 78-80].
[Hutt, 1977, p. 43], on textiles from Scandinavia (see Figure 4), Ancient Mexico [Weitlaner-Johnson, 1976, vol. 1, Plate 63], Nigeria, Algeria [Picton & Mack, 1979, pp. 35, 75] etc., on baskets from Lesotho, Mozambique, etc., and on a game board in Liberia [Machatscheck, 1984, p. 55].

Mat design from Angola

Figure 3

Scandinavian textile design

Figure 4
Discovering the Pythagorean theorem

Looking at the number of unit squares on each row of a “star” (see Figure 5), it is easy to see that the area of the “star” is equal to the sum of areas of the 4 x 4 shaded square and the 3 x 3 unshaded square.

![Diagram of a star figure]

A “star” figure, like the one in Figure 1, may also be called a “toothed square.” A toothed square, especially one with many teeth, looks almost like a real square. So naturally the following question can also be led in many other ways to draw this conclusion. E.g., the teacher may ask them to transform a “star” made of loose tiles into two mono-coloured similar figures. Or, one may ask them to cut off the biggest possible square from a “star” made of paper or cardboard and to analyse which figures can be laid down with the other pieces (see Figure 14).
arises: is it possible to transform a toothed square into a real square of the same area? By experimentation (see Figure 6), the pupils may be led to draw the conclusion that this is indeed possible.

![Figure 6](image)

In Figure 5, we have seen that the area of a toothed square (T) is equal to the sum of the areas of the two smaller squares (A and B):

$$T = A + B.$$  

In Figure 6 we conclude that the area of a toothed square (T) is equal to the area of a real square (C). Since $C = T$, we can conclude that

$$A + B = C.$$  

Do there exist other relationships between these three squares? What happens if one draws the toothed square and the two real squares (into which it is decomposed) together on square grid paper, in such a way that they become “neighbours”? Figure 7 shows a possible solution. When we now draw the last real square (area C) on the same figure, we arrive at the Pythagorean Theorem for the case of $(a, b, c)$ right triangles with $a : b = n : (n + 1)$, where the initial toothed square has $(n + 1)$ teeth on each side.
Figure 8 illustrates the Pythagorean Proposition for the special
case of the (3, 4, 5) right triangle. On the basis of these experiences, the pupils may be led to *conjecture* the Pythagorean theorem in general. In this manner, toothed squares assume a *heuristic value* for the discovery of this important proposition.

Does this same discovery process also suggest any (new) *demonstrations* for the Pythagorean theorem?

What happens when one reverses the process? When one begins with two arbitrary squares and uses them to generate a toothed square?

**A first proof**

Let $A'$ and $B'$ be two arbitrary squares. We look at Figure 5 for inspiration: dissect $A'$ into 9 little congruent squares, and $B'$ into 16 congruent squares, and join the 25 pieces together as in Figure 9. The obtained toothed square $T'$ is equal in area (T) to the sum of the real squares $A'$ and $B'$:

$$T = A + B.$$
Figure 9

Figure 10
As, once again, the toothed square is **easily transformed** into a real square $C$ of the same area (see Figure 10), we arrive at
\[ A + B = C, \]
i.e. the Pythagorean Proposition in all its generality.

![Diagram of a transformed square](image)

$n = 14$

Figure 11

**An infinity of proofs**

Instead of dissecting $A'$ and $B'$ into 9 and 16 subsquares, it is possible to dissect them into $n^2$ and $(n+1)^2$ congruent subsquares for each value of $n$ ($n \in \mathbb{N}$). Figure 11 illustrates the case $n = 14$. To each value of $n$ there corresponds a **proof** of the Pythagorean
Proposition. In other words, there exist infinitely many demonstrations of this famous theorem.

For relatively high values of $n$, the truth of the Pythagorean Proposition is almost immediately visible.

When we take the limit $n \rightarrow \infty$, we arrive at one more demonstration of the theorem.

For $n = 1$, one obtains a very short, easily understandable proof (see Figure 12).

These demonstrations were elaborated by the author in [1986d]. Another infinite set of possible dissection proofs for the same proposition has been outlined by Bernstein [1924].

Another proof by means of limits has been given in Gerdes [1986c].
Pappos’ theorem

Analogously, Pappos’ generalization of the Pythagorean Proposition for parallelograms can be proved in infinitely many ways (Figure 13 illustrates the case $n = 3$).

Figure 13
Example

Loomis’ well known study *The Pythagorean Proposition* gives “… in all 370 different proofs, each proof calling for its specific figure” [1940; 1972, p. 269] and its author invites his audience to “Read and take your choice; or better, find a new, a different proof …” [p. 13]. Our reflection on a widespread decorative motif led us not only to an alternative, active way to introduce the Pythagorean Proposition in the classroom but also to generate infinitely many proofs of the same theorem. May this example serve as a further stimulus to the multi-culturalisation of mathematics education.

![Alternative decomposition](image)

Alternative decomposition

Figure 14
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Chapter 7

‘PYTHAGORAS’, SIMILAR TRIANGLES AND THE “ELEPHANTS’-DEFENCE”-PATTERN OF THE (BA)KUBA (CENTRAL AFRICA) *

Introduction

The (Ba)Kuba people inhabit the central part of the Congo basin (in today’s Democratic Republic of Congo), living in the savannah south of the dense equatorial forest. The (Ba)Kuba had constituted a strong and secular kingdom.

Their metallic products, like weapons and jewellery are famous. The villages had specialized themselves in certain types of craft, like the production of ornamented wooden boxes and cups, velvet carpets, copper pipes, raffia cloths, etc. The beautiful decorative art of the (Ba)Kuba [see the examples in Figure 1 (Meurant, 1987, p. 170, 172)] attracted not only the attention of artists from all over the world (Meurant, 1987), but also from Mathematicians. D.Crowe (USA) analysed symmetry patterns occurring in (Ba)Kuba designs and found their mathematical variety and richness evident (Crowe, 1971; Zaslavsky, 1973, chap. 14). In this paper I shall present some (Ba)Kuba patterns and show how these designs may serve as an interesting starting point in the study of geometry.

A particular case of the Pythagorean proposition

(Ba)Kuba designs, like the one illustrated in Figure 2 (Meurant, p. 168) may serve to discover the Pythagorean proposition in the particular case where both legs are equal (see Figure 3).

The MONGO and MWOONG patterns

Figure 4 (Meurant, p. 114, 126, 144, 176) shows two tattooing motifs formerly used by the Ngongo, one of the ethnic groups that
belonged to the (Ba)Kuba kingdom. Both display a rotational symmetry of order 4.

When one draws lines between the ends of these tattooing motifs, one obtains a pattern similar to the (Ba)Kuba engravings illustrated in Figures 5 and 6 (Meurant, p. 176). The Ngongo call the engraving motif in Figure 5 *mongo*, i.e. knee (according to information collected by E.Torday in 1907. Cf. Meurant, 1987, p. 177). The (Bu)Shongo – the dominant ethnic group in the old (Ba)Kuba kingdom – call the engraving pattern in Figure 6a *mwoong*, i.e. “elephants’ defence”, and the motif in Figure 6b *ikwaakl’imwoong*, i.e. “deformed elephants’ defence” (Meurant, 1987, p. 177).
To arrive at the Pythagorean proposition

The (Ba)Kuba engraving pattern in Figure 6a is well known from the history of Mathematics in Asia (China, India). When one places several patterns together, one may discover, ‘reinvent’ and even prove, easily and geometrically, the so-called ‘Theorem of Pythagoras’, as Figure 7 illustrates. On the other hand, knowing the identity 

\[(b-a)^2 = b^2 - 2ab + a^2\]

and the formulas for the determination of the areas of squares and right triangles, one may arrive, in the following algebraic-geometric way, at the same conclusion. The area of the little central square is equal to \((b-a)^2\) and the areas of the four neighbouring right triangles (see Figure 8) are together \(2ab\). Therefore

\[c^2 = (b-a)^2 + 2ab = b^2 + a^2.\]
Chapter 7

Figure 7

Figure 8

Figure 9
An ornamental variant and a generalization of the Pythagorean proposition

Figure 9 shows a (Ba)Kuba variant of the elephants’ defence pattern. ¹ The rectangle is composed of two pairs of similar triangles (see Figure 10) and a little rectangle at the centre.

![Figure 10](image1)

Figure 10

![Figure 11](image2)

Figure 11

¹ During the 6th International Congress of Mathematical Education (August 1988), I had the opportunity of visiting the permanent exposition ‘From clan to civilization’ at the Ethnographic Museum of Budapest (Hungary). In Room VIII art and craft of the (Ba)Kuba is exhibited. The “elephants’ defence” pattern is seen on some exhibited objects. On the walls and top of a beautiful carved wooden box one encounters the referred variant of the same pattern.
When one places several examples of this variant together (see Figure 11), one may discover / invent and prove the following generalization of the Pythagorean proposition:

(2) \[ c' c = a' a + b' b \] (see Figure 12)

![Diagram of the Pythagorean proposition generalization](image)

Figure 12

On the other hand, it is possible to arrive at the same conclusion by an algebraic-geometrical reasoning. The area of the central rectangle is equal to \((b' - a)(b - a')\) and the areas of the four neighbouring right triangles are together \(ab + a'b'\). Therefore, we have

\[ c' c = (b' - a)(b - a') + ab + a'b' = a'a + b'b'. \]

**A combination of the Pythagorean Proposition with its generalization**

What result may be obtained when one combines the theorem (2)
(2) \[ c \ c' = a \ a' + b \ b' \]

with the Pythagorean proposition?

Combining (2) with the Theorem of Pythagoras

(1) \[ c^2 = a^2 + b^2 \]

and

(1') \[ (c')^2 = (a')^2 + (b')^2, \]

one obtains

(3) \[ (a \ a' + b \ b')^2 = (c \ c')^2 = c^2 (c')^2 = [a^2 + b^2][(a')^2 + (b')^2]. \]

This leads immediately to

(4) \[ 2 \ a \ a' \ b \ b' = a^2 (b')^2 + b^2 (a')^2 \]

and

(5) \[ (a \ b' - b \ a')^2 = 0. \]

In this way, we see that

(6) \[ a \ b' = b \ a', \]

i.e.

(7) \[ a : b = a' : b'. \]

In other words, the ratios of the legs (taken in the same order) of similar right triangles are equal. The deduction here presented is algebraic. Does there not exist a (purely) geometrical alternative?

A geometrical alternative

Let us return to the rectangular (Ba)Kuba variant of the elephants’ defence pattern (Figure 9). How can one join some of these ornamental rectangles in order to obtain a square?

Figures 13a and b show two possibilities to obtain a square of side \( c + c' \). In both cases, there appears a ‘hole’ in the centre whose sides measure \( c' - c \). When one extends the legs that end at the vertices of the square ‘hole’ until they encounter other legs, one obtains Figures 14a and b.
Figure 13
In both cases, there appear four right triangles around the square ‘hole’ (see Figure 15), that form together with the square ‘hole’ a new square of side $(a'+b') - (a+b)$. And in both cases, there appear around
this new central square four rectangles, which constitute patterns that are similar to the (Ba)Kuba tattooing displayed in Figure 4b.

As the big squares are congruent and the little squares are also congruent, one arrives at the conclusion that the patterns themselves (see Figure 16) are equal in area. Both patterns are composed of four rectangles. Therefore, the area \((ab')\) of one of these rectangles of the first pattern (Figure 16a) is equal to the area \((a'b)\) of one of the rectangles of the second pattern (Figure 16b):

\[
(8) \quad ab' = a'b.
\]

In other words

\[
(9) \quad a : b = a' : b',
\]
that is, we arrived geometrically at the conclusion that the ratios of the legs (taken in the same order) of similar right triangles are equal. It turns out to be easy to prove the Fundamental Theorem of Similar Triangles on the basis of the foregoing result. Also one may introduce without any difficulty the concepts of tangent, sine and cosine of an acute angle. Application of theorem (2) in the case of the triangles in Figure 17 leads to the trigonometric formula

\[(10) \quad c = a \sin \alpha + b \sin \beta.\]

\[\text{Figure 17}\]

**Other geometrical alternatives**

Inspired by the (Ba)Kuba designs, we saw that there exists a (purely) geometrical deduction of the theorem that says that the ratios of the legs (taken in the same order) of similar right triangles are equal. As in the schoolbooks we know this is either not proven or is proven on the basis of more general knowledge (Fundamental Theorem of Similar Triangles), I was led to look for other alternatives. I found the following: both may easily be used in the mathematics classroom, for instance, when the trigonometric ratios of an acute angle are introduced.

Let us consider a right triangle with sides \(a+a', b+b'\) and \(c+c'\) (Figure 18a). The right triangles of sides \(a, b, c\) and \(a', b', c'\) may fit into it in various ways. Figures 18b and c give two possibilities. The rectangles that remain have obviously the same areas. Therefore:

\[(8) \quad ab' = a'b.\]
Let us now consider right triangles with sides $a$, $b$, $c$ and $a'$, $b'$, $c'$. One may join them like illustrated in Figures 19a and b. In both cases, one may consider the so obtained figure as composed of a rectangle and a right triangle with sides $a' - a$, $b' - b$ and $c' - c$ (see Figure 20). Therefore, both rectangles have the same area, and one may conclude:

$$ab' = a'b.$$
Once more ‘Pythagoras’

The well-known drawing in Figure 15 constitutes an invitation for the discovery of some more proofs of the ‘Theorem of Pythagoras’.

Final considerations

(Ba)Kuba designs may serve as an attractive starting point in the mathematics classroom for the (re)discovery / invention and demonstration of the ‘Theorem of Pythagoras’, of one of its generalizations and of (a particular case of) the Fundamental Theorem of Similar Triangles. It was also shown how the considered (Ba)Kuba patterns may be used for the introduction of trigonometric ratios. Our reflection about the possibilities of incorporating cultural elements of the (Ba)Kuba into the teaching of geometry inspired us to find some didactical alternatives.

References


Chapter 8

ON POSSIBLE USES OF TRADITIONAL ANGOLAN SAND DRAWINGS IN THE MATHEMATICS CLASSROOM *

Abstract

Both industrialised and Third World countries see themselves confronted with the need to ‘multi-culturalise’ their mathematics curricula. Following a brief description of the drawing tradition of the Tchokwe people (Angola), some possible uses of their pictograms in the mathematics classroom are suggested. The examples given in this paper range from the study of arithmetical relationships, progressions, symmetry, similarity, and Euler graphs to the determination of the greatest common divisor of two natural numbers.

Introduction

In industrialised countries like Great Britain, France and U.S.A., the necessity to re-evaluate the total school experience in view of the educational ‘failure’ of many children from ethnic minority communities is more and more recognised. Pressure has mounted to reflect the multi-cultural nature of these societies in the school curriculum. Thus Bishop (1987, p. 2) concludes that it is necessary to ‘multi-culturalise’ the mathematics curriculum (see also Ginsburg and

* Published in the international journal Educational Studies in Mathematics (Dordrecht / Boston, 1988, Vol. 19, No. 1, 3-22, and reproduced in the Brazilian journal BOLEMA (São Paulo State University, Rio Claro, 1989, Special No. 1, pp. 51-78). A preliminary version of this article was published in the Nigerian journal Abacus (Ilorin, 1988, Vol. 18, No. 1, pp. 107-125).
As the inherited colonial boundaries seldom did justice to the existing cultural and ethnic realities, many Third World countries see themselves today, in the difficult process of nation building, confronted with the same need to multi-culturalise their mathematics curriculum (see e.g. D’Ambrosio, 1985a,b; Eshiwani, 1979; Gerdes, 1985b, 1986b; Nebres, 1983). During the colonial period, mathematics was generally presented as an exclusively white man’s creation and ability (cf. Gerdes, 1985a; Njock, 1985). With the hasty transplantation of curricula from the highly industrialised capitalist nations to Third World countries in the 1960s (e.g. the so-called ‘African Mathematics Program’), this negation of African, Asian, American-Indian and Aboriginal-Australian ‘indigenous’ mathematics continued, at least implicitly. And herein lies one of the fundamental causes for the verified low levels of attainment, which, in their turn, may reinforce (neo)colonial and racial prejudices (cf. D’Ambrosio, 1985b; Gerdes, 1985b).

In order to break this vicious circle, D’Ambrosio (1985a,b) stresses that it is urgent to recognise all sorts of ‘indigenous’ mathematics (or ‘ethno-mathematics’) and to integrate / incorporate them into the curriculum. Only then, in the author’s opinion, is one necessary condition for ‘world-mathematics’, (defined as the union of all ‘ethno-mathematics’; terminology of Hogbe-Nlend, 1985) or ‘internationalised mathematics’ (terminology of Bishop, 1987) to become really and effectively accessible to the peoples of the Third World satisfied.

The following article on traditional Angolan sand drawings is meant as a concrete example of how it is possible to use ‘indigenous’ mathematical ideas in the teaching context. We are looking for effective bridges (Gay and Cole, 1967, p. 94) between ‘ethno-mathematics’ and ‘world-mathematics’.

The drawing tradition of the Tchokwe

The Tchokwe people (or Cokwe or Quiocos in Portuguese) with a population of about one million (Fontinha, 1983, p. 28) inhabit predominantly the northeast of Angola, the Lunda region.
Traditionally they are hunters, but since the middle of the 17th century they have dedicated themselves also to agriculture (Redinha, 1975, p. 11). The Tchokwe are well known for their beautiful decorative art (Bastin, 1961), ranging from the ornamentation of plaited mats and baskets, iron work, ceramics, engraved calabash fruits (Fontinha and Videira, 1963) and tattoos (Lima, 1956) to paintings on house walls (Redinha, 1953) and sand drawings (Fontinha, 1983).

When the Tchokwe meet at their central village places or at their hunting camps, they are used, sitting around a fire or in the shadow of leafy trees, to spend their time in conversations that are illustrated by drawings on the ground. Most of these drawings belong to a long tradition. They refer to proverbs, fables, games, riddles, animals etc. and play an important role in the transmission of knowledge and wisdom from one generation to the next (Fontinha, 1983, p. 37). The designs have to be executed smoothly and continuously, as any hesitation or stopping on the part of the drawer is interpreted by the audience as an imperfection and lack of knowledge, and assented with an ironic smile.

In order to facilitate the memorisation of their standardised picto- or ideograms, the ‘akwa kuta sona’ – drawing experts – invented an interesting mnemonic device. After cleaning and smoothing the ground, they first set out with their fingertips an orthogonal net of equidistant points. The number of rows and columns depends on the motif to be represented. For example, in order to represent an antelope, one needs three rows of four points (Figure 1). By applying their method – an example of an early use of a coordinate system! (cf. Santos, 1960, p. 267) – the ‘akwa kuta sona’ generally reduce the memorisation of a whole design to that of mostly two numbers and a geometric algorithm.


Some possible uses of these traditional Tchokwe sand drawings in the mathematics classroom will now be suggested and briefly discussed.
Figure 1

5 x 5

\[ a = 4 \cdot x + 1 \]

Figure 2
Arithmetical relationships

Each drawing redistributes, we may say, the points of the reference net. This situation may be “exploited” in the mathematics classroom, as the following examples will show, to discover diverse arithmetical relationships.

First example

On the basis of the representation of a forest with many ‘qundu’ birds (Figure 2b) one finds

\[ 5^2 = 4 \times 6 + 1. \]

Arithmetical progressions

An analysis of the symbolic representation of the “chased chicken” pattern (Figure 3) leads to

\[ 5 \times 6 = (1+2+3+4+5) + (5+4+3+2+1) = 2 \times (1+2+3+4+5). \]
This and other related examples can be used as a starting point for the study of sums of arithmetical progressions

\[ 2 \times (1 + 2 + 3 + ... + n) = n (n + 1) \text{ etc.}, \]
or, alternatively

\[ 2 \times (1 + 2 + 3 + n) = (n + 1)^2 - (n + 1), \]
as can be extrapolated from

\[ 2 \times (1 + 2 + 3 + 4 + 5) = 6^2 - 6, \]
as suggested by a representation of ‘Kalunga’ (God) [see Figure 4c].

In order to execute many Tchokwe sand drawings, it is necessary to superimpose two orthogonal nets in such a way that the points of the second one are the centres of the unit squares of the first net, thus forming together a new grid. The representation of a tortoise illustrates this superimposition (Figure 5). The redistribution of reference points by this tortoise figure suggests:

\[ 3^2 + 2^2 = 1+3+5+3+1. \]

In the same way, the representation of an ox-stall and of an elephant’s head (Figures 6b, 7b) lead to:

\[ 4^2 + 3^2 = 1+3+5+7+5+3+1 \]
and

\[ 5^2 + 4^2 = 1+3+5+7+9+7+5+3+1. \]

Which part of \((1 + 3 + 5 + 7 + 5 + 3 + 1)\) corresponds to \(4^2\)? Which part to \(3^2\)? By experimentation, one may observe, e.g. in the following way (Figure 8a):

\[ 4^2 = 3^2 + 3 + 4, \text{ i.e. } 4^2 = 3^2 + 7. \]

Therefore \(4^2 = 1+3+5+7\) and \(3^2 = 5+3+1\). One may ask the students if \(5^2 + 4^2\) or \((1+3+5+7+9+7+5+3+1)\) can be split up in the same manner. And: “Are there other ways to see that \(4^2 = 1+3+5+7\), starting with the \(4^2\) point square grid?” (Figure 8b).
Figure 4
Figure 4
Figure 5

3^2 + 2^2

Figure 6

4^2 + 3^2
Extrapolation gives
\[ n^2 = 1 + 3 + 5 + ... + (2n - 1), \]
or the sum of the first \( n \) odd numbers is \( n^2 \).
A Pythagorean Triplet

A 5 point reference frame is used for the characteristic ‘cingelyengelye’ motif (Figure 9), a very old design that appears already on rock paintings in the Upper Zambeze region (Redinha, 1948) and for some other Tchokwe patterns (Figure 10; Bastin, 1961, 152; Fontinha, 1983, 213, 207, 239, 227; Redinha, 1948, 74). By trying to cover double square reference frames \([n^2 + (n + 1)^2\] points\) with these patterns, pupils may discover the ‘Pythagorean’ triplet (3, 4, 5):

\[3^2 + 4^2 = 5^2\] (Figure 11).
Geometrical ideas

The tradition of the ‘akwa kuta sona’ reveals a profound awareness and interest in the geometrical properties of their drawings. These properties, such as symmetries and similarities, are studied in the classroom and the sand drawings may serve as a starting point as the following examples will show.

Figure 12

Axial Symmetry

Not only the final sand drawings (Figure 12) display a bilateral symmetry, but also the reference frames themselves. Two corresponding points have the same distance to the axis of symmetry.

Double Bilateral and Point Symmetry

Double bilateral and point symmetries are displayed not only by the final sand drawings (Figure 13), but also by the orthogonal reference nets themselves. This may help the pupils to discover that corresponding points have the same distance to the centre of symmetry.
Figure 13

Figure 14
Rotational Symmetry

In the same way, pupils may be led to the discovery (Figure 14), that corresponding points are at the same distance from the centre of rotation.

Similarity

Figure 15e represents a lioness with her two cubs. The dimensions $10:3$ and $7:2$ of the rectangular skeletons of the lioness and her cubs have been chosen in such a way that the lioness and the cubs are more or less similar figures. Therefore

$$10 : 3 \approx 7 : 2.$$ 

Their mutual orthogonal position may be used in the mathematics classroom to compare (see Figure 16) the pairs $(3, 7)$ and $(2,10)$ and discover that

$$10 : 3 \approx 7 : 2$$

corresponds to

$$3 \times 7 \approx 10 \times 2 \quad (21 \approx 20).$$

An analogous analysis (Figure 17) of the representation of a leopard with five cubs leads to the correspondence of

$$11 : 8 \approx 4 : 3$$

and

$$8 \times 4 \approx 11 \times 3 \quad (32 \approx 33).$$

By extrapolation one gets:

$$a : b \approx c : d \quad \text{or} \quad a : b = c : d$$

is equivalent to

$$b \times c \approx a \times d \quad \text{or} \quad b \times c = a \times d.$$
Figure 15 (first part)
Figure 15 (Second part)

Figure 16
Let us return to the representation of a tortoise (see Figure 5). The ‘akwa kuta sona’ started with a $3 \times 3$ reference frame and needed essentially three closed curves to complete their drawing (Figure 18). In the case of the elephant’s head of Figure 6, one starts with a $5 \times 5$ reference frame and needs five closed curves to “embrace” all the points.

Figure 17

*Geometrical Determination of the Greatest Common Divisor of two Natural Numbers*
To draw an antelope’s head (Fontinha, 1983, 235), one has to start with a $2 \times 4$ frame and needs two closed curves to “embrace” all the network points (Figure 19).

How many such curves are necessary to “embrace” all points of a $m \times n$ reference frame?

First of all, the pupils may try to define the characteristics of these curves: they form both before and after a reflection, angles of $45^0$ with the sides of the reference frame (see Figure 20). Now the
pupils may experiment (Figure 21) and discover that it is only necessary to draw one such closed curve in order to answer our question:

\[ f(3, 6) = ? \]

Figure 21

An admitted closed curve that starts from a vertex (see Figure 22a) always “embraces” one point of each column and two of each row. As there are three rows, three curves are necessary: \( f(3, 6) = 3 \).

\[ f(4, 6) = ? \]

This time, an admitted closed curve (see Figure 22b) “embraces” two points of each initial six columns and three of each row. As there are four rows, \( \frac{4}{2} = 2 \) curves are necessary. In the same way, as there are 6 columns, \( \frac{6}{3} = 2 \) curves are necessary.

Figure 22
Extrapolation leads to

\[ f(m, n) \text{ is a divisor of } m, \]
\[ f(m, n) \text{ is a divisor of } n, \]
or \( f(m, n) \) is a common divisor of \( m \) and \( n \). Further experience leads to: \( f(m, n) \) is the greatest common divisor of \( m \) and \( n \) \[= \text{gcd}(m, n)\].

![Figure 23](image_url)

Figure 23 gives an application of this provisional result. Cautious observation of the behaviour of the closed curve drawn in Figure 23 and comparison with other concrete examples may lead the pupils to discover the following interesting generalisation:

\[ \text{gcd}(m, n) = \text{minimum number of points “embraced” by “branches” of an admitted closed curve that passes through a } m \times n \text{ reference frame}. \]
Towards the Euclidean algorithm

From this generalisation onwards, it is not very difficult for the pupils to arrive by means of a structured series of exercises (see Gerdes, 1987a) at the geometric equivalent of Euclid’s arithmetical algorithm for the determination of the greatest common divisor of two natural numbers. Figure 24 illustrates the possible successive steps in this discovery process for the case \((m, n) = (21, 15)\). In Figure 24b, the admitted closed curve has been substituted by a polygonal line. At this stage, the \(\text{gcd}(m, n)\) may be geometrically interpreted in the following way (cf. Figure 24c):

\[
\text{gcd}(m, n) = \text{length of the side of the greatest square with which it is possible to fill up a } \text{m} \times \text{n} \text{ rectangle (same units of length)}. 
\]

Figure 24d shows that it is not necessary to draw the whole polygonal line of 24b in order to find \(\text{gcd}(m, n)\). It is sufficient to consider a reduced polygonal line, that “diagonally cuts off squares” from the original \(m \times n\) rectangle.

Euler Graphs

The Tchokwe sand drawing tradition offers good possibilities to study some properties of graphs. For example, one may ask the pupils which figures can be drawn in the sand without lifting the finger or retracting any line segment (cf. Zaslavsky, 1973, p. 105).

The representations of a ‘dyahotwa’ bird (see Figure 25a, b; Fontinha, 1983, 149, 151) are not traceable. But they may be ‘extended’ in order to become so (Figure 25c, d). The pupils may discover that there exists a path that traverses the figure without crossing any segment more than once if and only if there are fewer than 3 vertices that belong to an odd number of segments. The representation of a married couple (Figure 26; Fontinha, 1983, 145) is an example of such an Euler graph.
Figure 24
Figure 24
Figure 24
Figure 24
Concluding remarks

In this paper we suggested some possible ways to use traditional Angolan sand drawings in the mathematics classroom. The incorporation of this Tchokwe, both educational and artistic-mathematical, tradition into the curriculum may contribute to achieve
important societal goals:

* in the concrete case of the Tchokwe people it may contribute to the *revival*, reinforcement and *valuing* of this practice of the ‘akwa kuta sona’, threatened with extinction during the colonial occupation; it may contribute towards a more productive and creative mathematics education, as it avoids sociocultural and psychological alienation;

* with the integration of this regional tradition into the *national* curriculum, this drawing practice, the knowledge it reveals and its *mathematical potential*, will become less monopolised, less regional, less linked to a particular ethnic group; the incorporation of this and other popular practices from all regions of the country will contribute to the development of a truly national culture, very important in a process of nation building as in the case of Angola (cf. Gerdes, 1986b);

* the uses of these sand drawing patterns in the mathematics classroom need not be restricted to Angola. On the contrary. Their incorporation in other African curricula, for example in Mozambique, will contribute to the valuing and appreciation of the culture of this brother people, will reinforce the apprehension of the value of the artistic and scientific heritage of the African continent; will consolidate the idea that mathematics does not come from outside the African cultures. In other societies as well (cf. Zaslavsky, 1973, 1979, 1985) their ‘*multi-culturalising*’ integration into the curriculum may stimulate interdisciplinarity (e.g. mathematics and drawing with artistic-aesthetic education) and contribute to mathematical confidence: mathematics is *panhuman*; *all* peoples have been (and are) capable of developing mathematics (‘ethnomathematics’). This confidence will facilitate the assimilation of ‘worldmathematics’.

Some other examples of incorporating traditional practices into the mathematics curriculum are given in Gerdes (1986a, 1987b).
Postscript

In this paper we explored some possibilities of using the Tchowke sand drawing tradition into the mathematics classroom. Nevertheless their educational value is not the only reason to study them. They also constitute a stimulus for the development of new mathematical ideas and methods (see e.g. the results of Jaritz, 1983) and are very important for the reconstruction of some parts of the early history of mathematics. Probably independent of the Tchokwe tradition, similar drawing methods have been developed in South India (Layard, 1937) and on the New Hebrides (Deacon, 1934; Ascher, forthcoming). It seems that different people invented the same mnemonic device in response to the same need of facilitating the transmission of ideas and values from one generation to the next. There are reasons to believe (see e.g. the figures in Giacardi et al., 1979, 146, 157) that in Ancient Mesopotamia a similar tradition existed. The author is preparing a study on the possible influence of a tradition of this type of Tchokwe sand drawings on the development of Ancient Mesopotamian mathematics, one important root of ‘world-mathematics’.

Acknowledgements

The author is grateful to Professors D. Crowe (Madison, U.S.A.), M. Ascher and C. Zaslavsky (New York, U.S.A.) and to the editors for their valuable comments on this paper and to Dr. W. Humbane (Maputo, Mozambique) for proofreading.

Sources for illustrations

Figure 1: drawn after Fontinha, 1983, p. 187;
Figure 2b: drawn after Fontinha, 1983, p. 271;
Figure 3b: drawn after Fontinha, 1983, p. 287;
Figure 4c: drawn after Fontinha, 1983, p. 255, 177; cf. Zaslavsky, 1973, p. 109;
Figure 5e: drawn after Fontinha, 1983, p. 221;
Figures 6b, 7b: drawn after Fontinha, 1983, p. 167;
Figure 12: drawn after Fontinha, 1983, 229, 211, 197, 171;
Figure 13: drawn after Fontinha, 1983, 181, 257, 133, 185, 189, 195, 245;
Figure 14: drawn after Fontinha, 1983, 243, 263, 175, 183;
Figure 15e: drawn after Fontinha, 1983, p. 183;
Figure 19b: drawn after Fontinha, 1983, p. 235;
Figure 25a, b: drawn after Fontinha, 1983, p. 149, 151;
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Chapter 9

EXPLORATION OF THE MATHEMATICAL POTENTIAL OF ‘SONA’:
AN EXAMPLE OF STIMULATING CULTURAL AWARENESS IN MATHEMATICS TEACHER EDUCATION *

Introduction: necessity of culture-oriented education

Development strategies that neglect or minimize cultural factors provoke indifference, alienation and social discord, as underlines the Report of the South Commission, led by the former President of Tanzania, Julius Nyerere. 1 Alternate development strategies should utilize the enormous reserves of traditional knowledge, creativity and capacity to take initiatives that exist in the Third World (p.55). The Regional Consultation on Education for All (Dakar, 27-30 November 1989) stresses that Africa needs culture-oriented education. Scientific appreciation of African cultural elements and experience is considered to be “one sure way of getting Africans to see science as a


means of understanding their cultures and as a tool to serve and advance their cultures” (p.23).

_Educate or Perish: Africa’s Impasse and Prospects_, study directed by Joseph Ki-Zerbo, ³ shows that today’s African educational system – unadapted and elitist – favours foreign consumption without generating a culture that is both compatible with the original civilization and truly promising. Africa needs a “new educational system, properly rooted in both society and environment, and therefore apt to generate the self-confidence from which imagination springs.” (p.104).

**Ethnomathematical research and teacher education**

In order to avoid alienation – the experience of mathematics as a rather strange, useless and uninteresting subject, imported from outside Africa – the African mathematical heritage, ⁴ traditions and practices have to be ‘embedded’ or ‘incorporated’ into the curriculum.

In order to prepare gradually a curriculum reform that guarantees that mathematics education is “in tune with African traditions and socio-cultural environment,” ⁵ we started ethnomathematical research in Mozambique.

_Ethnomathematical_ studies analyse ⁶


* mathematical traditions, that survived colonization and mathematical activities in people’s daily life and ways to incorporate them into the curriculum;

* cultural elements that may serve as a starting point for doing and elaborating mathematics in and outside school.

As teachers fulfil a pivotal role in any (successful) curriculum reform, teacher education is a strategic place for debate and experimentation with ‘cultural embedding of mathematics education.’

In my paper, ‘On culture, geometrical thinking and mathematics education’, (1988) 7 examples of ‘cultural conscientisation’ of future mathematics teachers are given: a Mozambican house construction and a study of alternate axiomatic constructions of Euclidean geometry; funnel weaving as a source of inspiration for a general method for the construction of regular polygons; from button weaving to the ‘Theorem of Pythagoras;’ from traditional fish traps to an alternate circular function, tilings and the generation of semi-regular and regular polyhedra. This time I should like to expose how investigations by future mathematics teachers may be realized examining the cultural context and history of Southern Africa, opening new field for mathematical research. As an illustration, the mathematical potential of *sona*, sand drawings or ‘sand graphs’ of the Tchokwe people of North-eastern Angola will be explored.

**The (lu)sona tradition**

The *(lu)sona*-tradition belongs to the heritage of the Tchokwe, Lunda, Lwena, Xinge and Minungo peoples, that inhabit the north-
eastern part of Angola, and of the Ngangela and Luchazi peoples of south-eastern Angola and north-western Zambia. Especially the Tchokwe are well known for their beautiful decorative art, ranging from the ornamentation of plaited mats and baskets, iron work, ceramics, engraved calabash fruits and tattoos to paintings on house walls and sand drawings.

When the Tchokwe met at their central village places or at their hunting camps, they usually sat around a fire or in the shadow of leafy trees spending their time in conversations, illustrated by drawings in the sand. These drawings are called *lusona* (singular) or *sona* (plural).

Most of these drawings belong to an old tradition. They refer to proverbs, fables, riddles, animals, etc. and played an important role in the transmission of knowledge and wisdom from one generation to the next.

Every boy learned the meaning and execution of the easier *sona* during the intensive schooling phase of the circumcision and initiation rites. The meaning and execution of more difficult *sona* was only known by specialists, the *akwa kuta sona* (those who know how to draw), who transmitted their knowledge to their sons.

The designs have to be executed smoothly and continuously, as the audience interpreted any hesitation or stopping on the part of the drawer as an imperfection and lack of knowledge.

In order to facilitate the memorisation of their standardised *sona*, the *akwa kuta sona* invented an interesting mnemonic device. After cleaning and smoothing the ground, they first set out with their fingertips a net of equidistant points. The number of rows and columns

![Figure 1](image-url)
depends on the motif to be represented. Then they draw the line Figure applying the geometrical algorithm that corresponds to the motif to be represented. Figure 1 displays an example of the representation of a bird.

**Didactical experimentation**

In the paper ‘On possible uses of traditional Angolan sand drawings in the mathematics classroom’ (1988), some possible uses of the *sona* are suggested to mathematics teachers. The examples given range from the study of arithmetical relationships, progressions, symmetry, similarity, and Euler graphs to the determination of the greatest common divisor of two natural numbers. I have also elaborated mathematical connections to *sona* in a book for children (age 10-15), *Living mathematics: drawings of Africa* (1990). In the paper ‘Find the missing figures’ (1988) and in the book *Lusona: Geometrical Recreations of Africa* (1991) mathematical amusements are presented that are inspired by the geometry of the sand drawing

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tradition. The problems included in these publications have been tested out in a seminar of teacher educators and of future mathematics teachers at Mozambique’s Higher Pedagogical Institute. Participation in the seminar was voluntary and normally it was difficult to finish the two-hour session of the seminar as participants ‘lost their notion of time’. The first type of *sona*-inspired-recreation consists of problems where some Figures of a series are given and one has to find the missing ones. Figure 2 gives an example.

![Figure 2](image_url)
Here all figures have to be \textit{monolinear}, i.e. made out of only one closed line that embraces all points of the reference frame without repeating itself. In order to find the solutions, both the dimensions of the successive drawings and the geometrical algorithm have to be carefully observed.


\textbf{Further exploration of the mathematical potential of SONA in teacher education}

The context of the \textit{sona} is very rich for further mathematical exploration. In general, it is important for future mathematics teachers to get a feeling of what it means to do mathematics: experiment, discover and formulate of conjecture, prove theorems. The study of \textit{sona} and \textit{sona}-like drawings gives a culturally embedded, attractive context in which to develop that feeling, as the following examples will show.

\textit{Composition rules}

Very apt for teacher-student investigations are the discovery and proof of the (reconstructed) traditional Tchokwe rules for the building up of bigger monolinear drawings out of smaller monolinear ones. Figure 3 illustrates one such a rule that has been applied four times in the Tchokwe representation of a leopard with five cubs (see Figure 4b).
Figure 3

Composition rule

Figure 4

Systematic construction of monolinear drawings

Figure 5 displays a left half of a *lusona*. It may be thought of as constructed on the basis of the ‘triangular design’ in Figure 6a by linking and passing through the endpoints $a$, $b$, $c$, $d$, $A$, $B$, $C$ and $D$ in the order $DaAbBcCd$. Many questions for further reflection and analysis arise, such as:
Do there exist other possibilities? (See Figure 6b) How many?

How many possibilities for monolinear drawings exist if there are \( n \) endpoints on each side of the ‘triangular design’?

How many of them are symmetrical like the lusona in Figure 5?
How many lines?

First example

The Tchokwe *lusona* that represents the *stomach of a lion* displays dimensions 5x4 and is monolinear (see Figure 7a). The monolinear Ngangela sand graph in Figure 7b represents a *bark blanket*. The same algorithm (illustrated in Figure 7c) has been applied. Only the dimensions of the grid are different: 5x6. When one uses the algorithm in the case of a reference frame of dimensions 7x3, one needs three lines in order to embrace all points of the grid (see Figure 8). How does the number of lines depend on the dimensions \( m \) and \( n \) of the grid?
When the width \( (m) \) is even, the drawing does not look like a lions’ stomach (see Figure 9). We consider therefore only odd values of \( m \). For the same reason \( n \) has to be equal to or greater than 2.

The teacher-students may experiment with concrete values of \( m \) and \( n \), draw the figures, count the lines and join the data collected in this manner in a table like:

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The extrapolation on the basis of these experimental data may lead to the following table:

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and to the formulation of a conjecture such as:

The number of lines in a ‘lion’s-stomach’ necessary to embrace all points of a $m \times n$ grid is equal to 1 if $n = 4p+1$ and equal to $n$ if $n \neq 4p+1$, where $p$ denotes a positive integer.

Now the teacher-students may test their conjecture. For instance, is the hypothesis verified in the case $m=4$, $n=13$ (see the monolinear drawing in Figure 10)?

![Figure 10](image)

The next question is: How to prove the conjecture?
Second example

Among the *sona* reported from the Tchokwe, there are two that represent the marks left on the ground by a chicken when it is chased. Both line drawings are monolinear and satisfy the same geometrical algorithm (see Figure 11a). The dimensions of the reference frames, however, are different: $5 \times 6$ and $9 \times 10$ (see Figure 11b and c). The monolinear sand drawing represented in Figure 11d has dimensions $8 \times 3$. It has been reported from the Ngangela. The graph shows the path of the ‘*ngonge*’ insect as it slowly eats its way around a tree, cutting out the flow of sap.

![Figure 11](image)

When one uses a reference frame of dimensions $5 \times 10$ and applies the same algorithm, one needs three of such ‘chased-chicken’ lines to embrace all points of the reference frame (see Figure 12).
Teacher-students may investigate the question: How does the number of ‘chased-chicken’ lines depend on the dimensions of the grid of points?

In order to get a design that looks like Figure 11, the width of the reference frame, i.e. the number of dots in a row, has to be even, and its height, i.e. the number of points in a column, has to be odd. The teacher-students may now experiment with concrete values for the width and height, draw the figures, count the number of ‘chased-chicken’ lines and join the data collected in this manner in a table like:

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Let $2m+1$ denote the height and $2n$ the width of the rectangular grid, where $m$ and $n$ are positive integers. Using $m$ and $n$ as variables, the table becomes easier to analyse:
The number $G$ of ‘chased-chicken’ lines that are needed to embrace all points of a reference frame is a function of the width and the height of the grid:

$$G = f(2m+1, 2n).$$

When one looks at the table and does not take into account the row $m=1$, it seems that corresponding rows and columns are equal, i.e.

$$f(2m+1, 2n) = f(2n+1, 2m).$$

Furthermore, (at least) the first rows seem to be periodic. In the first row, $(1, 2)$ repeats itself (period = 2); in the second row, $(3, 1, 1)$ seems to repeat (period = 3). The period seems to be equal to the row number $m$ plus one, i.e. period = $m+1$. On the diagonal, i.e. for $m=n$, one expects $f(2m+1, 2m) = m+1$.

Extrapolation on the basis of the collected experimental data and of the observed regularities, may lead to the following table:

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Now the teacher-students may test their conjecture. For instance, is the hypothesis verified in the case where \( m = 7, n = 9 \). Analysing such questions as:

* Is it possible to complete the table up to \( m = n = 20 \)?:

* What relationship does there seem to exist between the numbers in the third row and its period?

* What happens in the case of the fourth row? What happens in general?,

the teacher-students may discover that the numbers in any row or column are divisors of the respective period. And as any number in the table belongs at the same time to a column and to a row, it is a divisor of both periods: of the period (= \( m+1 \)) of the corresponding row and of the period (= \( n+1 \)) of the respective column. Therefore it may be conjectured that in general

\[
f(2m+1, 2n) \text{ is a common divisor of } m+1 \text{ and } n+1,
\]
or still more that

\[
f(2m+1, 2n) \text{ is the greatest common divisor (gcd) of } m+1 \text{ and } n+1.
\]

In other words the teacher-students arrive at the conjecture that

\[
f(2m+1, 2n) = \gcd (m+1, n+1).
\]

Once more, the next question is: How to prove the conjecture?
As variations on this theme, other challenging research questions arise like
* What characterizes (non)rectangular grids of monolinear ‘chased-chicken’ patterns? In other words, under what conditions is a ‘chased-chicken’ pattern monolinear? (see Figure 13 for two examples)

* Figure 14 shows a variant of the ‘chased-chicken’ pattern where
the successive vertical broad zigzags of the ‘chased-chicken’-line are (bilateral) symmetrical. Under what conditions are such patterns monolinear? What will be the genus of such patterns?

*Underlying black-and-white designs*

If one draws a monolinear *lusona*, e.g. the one shown in Figure 15, on squared paper and colours the squares through which the curve successively passes, alternately black and white, an underlying black-and-white design is obtained. Figure 16 shows the underlying black-and-white design of the *lusona* in Figure 15. Figure 17 displays another *lusona* with the same type of chequer-board as underlying black-and-white design.

![Figure 15](image_url)
Here arises a first research question:

* What is a necessary and sufficient condition for a monolinear drawing to produce an underlying chequer-board design?

Figure 18 shows two *sona* that generate black-and-white designs, which are different from chequer-boards.

* Which relationships exist between *sona*-type drawings and their underlying black-and-white designs?

* Which similarities exist between monolinear drawings, which generate the same underlying black-and-white design? Figure 19 gives an example. When one maintains the algorithms but
changes the dimensions of the drawings, do their underlying black-and-white designs continue to be the same?

A design is called a *two-colour design* if there is some rigid motion, which interchanges the colours everywhere. Figure 20 shows a
monolinear *sona*-type drawing and its underlying black-and-white design.

![Diagram of sona design](image)

Figure 20

This black-and-white design is a two-colour design, as a half-turn (rotation of $180^\circ$) reverses the colours. The (global) two-colour,
two-fold rotational symmetry of the black-and-white design corresponds to the (global) two-fold symmetry of the drawing. Figure 21 shows an (repeated) element of the drawing and some of its underlying black-and-white modules. The element displays a double symmetry. The black-and-white modules, on the other hand, have a (one-color) two-fold symmetry. In general:

* How do the symmetries of monolinear *sona*-type drawings relate to the local and global one-color and two-colour symmetries of the underlying black-and-white designs?

![Figure 21](image)

Another interesting research question is the following:

* What happens with the underlying two-colour designs when one changes the dimensions of the monolinear drawings?

Figure 22 illustrates an example, with successive dimensions 6 × 5, 10 × 9, 14 × 13 and 18 × 17.

* Is it possible to predict what will be the black-and-white designs in the cases 22 × 21 and 26 × 25?

* Is it possible to conjecture what will be the underlying black-and-white design for dimensions \((4n+2) \times (4n+1)\) of the drawing, where \(n\) denotes a positive integer?

And, much more general,
* Given the algorithm of a monolinear *sona*-type drawing and the dimensions of the reference frame, is it possible to predict what will be the underlying black-and-white design?

* Is it possible to characterise the class of black-and-white designs, which underlie (are generated by) *sona*-like drawings?

Once the conjectures are found, there remains the question of testing and proving them.

**Concluding remarks**

In this paper I described some experiences in a seminar I organized for teacher educators and future mathematics teachers. They reacted enthusiastically to the topic and were conscious that they were doing real mathematics. My experience suggest that by discovering both the didactical and the mathematical potential of a tradition like that of the *sona*, that belongs to the cultural heritage of Africa, future teachers gain confidence not only in their personal capacities and capacities as a group to do, to invent and to understand mathematics but also in the potentialities of African culture in general. In other words, one stimulates cultural awareness among future mathematics teachers, thereby satisfying one of the necessary conditions for mathematics education to become culture-oriented and emancipatory.


Figure 22
Figure 22
Figure 22

b2

c2
Figure 22

d1
Chapter 10

TECHNOLOGY, ART, GAMES AND MATHEMATICS EDUCATION: AN EXAMPLE *

Observe this type of handbag, common in Mozambique (Figure 1). These bags are made from three superimposed cylinders. One of the cylindrical walls of each of them has been eliminated by knitting two halves of one circular border together. The artisans usually decorate these bags with decorative strip patterns.

Handbag

Figure 1

* One of the examples given during the opening plenary address “Some Reflections on Mathematics Education and ‘The Challenge to the South’” at the 4th Brazilian National Meeting on Mathematics Education, January 26-31, 1992, Blumenau SC, Brazil [for more examples and details, see the books Sipatsi: Technology, Art and Geometry in Inhambane (co-author Gildo Bulafo), 1994, and Sipatsi: Basketry and Geometry in the Tonga Culture of Inhambane (Mozambique, Africa), 2009].
* What has the basket maker to do in order to guarantee that the bag is beautiful? What does he/she have to do so that the decorative motif repeats itself an integral number of times on the cylindrical wall without leaving any undecorated space or any motif incomplete?

Which calculations has she or he to make?

The total number of strands has to be an integral multiple of the number of strands necessary to produce the motif once. All strip patterns are symmetrical and many display a rotational symmetry of 180°. In other words, it does not make any difference if one looks at the strip pattern from above or from below; the impression one gets remains the same. Figure 2 gives examples of this type of decoration.

![Woven strip patterns](image)

Many questions arise:

* Is it possible to invent other decorations in the same style?
In which other contexts is it possible to observe cylinders or related forms with the same type of symmetrical decoration?

In pottery ... Figure 3 shows symmetric patterns used by women at a ceramics cooperative in a suburb of Maputo to decorate their pots, dishes and vases.

Decorative strip patterns on ceramics

Figure 3

In games... Many Mozambican children like to run with car tyres (Figure 4). Two sticks are put into a can. The base of the can slides in soapy water or in oil, contained in the bottom of the tyre. The children have discovered

* at which angle one has to push the sticks in order to run faster;
* at which angle one has to push the sticks to be able to brake or stop;
* how to hold the sticks to be able to make a curve or to turn round, etc.
There are many other games with tyres. Figure 5 gives examples. In the last case the tyre is used as a trampoline.
Children playing with tyres
(Drawings by Marcos Cherinda)

Figure 5

Tyre profiles
Figure 6
Looking at the tyres themselves, what particularity related to their profile may be observed?

The tyre profiles often display the same symmetry as the decoration on the handbags and pots (see the examples in Figure 6). Why does this happen?

* Is it possible to invent new profiles? Which are the best ones? In what respect? Why?
* In which other contexts, is it possible to meet patterns with the same type of rotational symmetry? Tattooing ... (see Figure 7).
* Do these symmetries also occur in nature?

![Tattoo design with half turn symmetry](Makonde, North Mozambique)

Figure 7
Chapter 11
On Mathematics in the History of
Africa South of the Sahara *

INTRODUCTION

In her classical study “Africa Counts: number and pattern in African culture” [1973a] (review in [Wilder 1976]), Claudia Zaslavsky presented an overview of the available literature on mathematics in the history of Sub-Saharan Africa. She discussed written, spoken and gesture counting, number symbolism, concepts of time, numbers and money, weights and measures, record-keeping (sticks and strings), mathematical games, magic squares, graphs, and geometric forms, while Donald Crowe contributed a chapter on geometric symmetries in African art.

Since the publication of Zaslavsky’s overview, many scholars, students, teachers and laymen alike – both in Africa and abroad – have become interested in the mathematical heritage of Sub-Saharan Africa. The African Mathematical Union (AMU) formed its Commission on the History of Mathematics in Africa (AMUCHMA) in 1986 in order to stimulate research in the history of mathematics in Africa in general, and to promote the dissemination of research findings and the exchange of information in this field. In 1987 the AMUCHMA began publishing a newsletter in English, French and Arabic.

In this paper, an overview of research findings and of sources on or related to the history of mathematics in Africa south of the Sahara is presented, which includes studies that have appeared since the publication of Africa Counts. Topics like counting and numeration systems, numerology, mathematical games and puzzles, geometry, graphs, Islam and mathematical development, international connections, and the history of mathematics curricula will be included. Attention will also be paid to the objectives of research in the history of mathematics in Africa, to methodology, to the relationship with ethnomathematical research and to the uses of research findings in mathematics education. Some possible directions for further research are also identified.
WHY STUDY THE HISTORY OF MATHEMATICS IN AFRICA SOUTH OF THE SAHARA?

There are many reasons, which make the general study of the history of mathematics both necessary and attractive (see e.g. [Struik 1980]). One of the most important reasons – and certainly valid in the case of Africa – is that it “helps to understand [the] cultural heritage, not only through the applications mathematics has had and still has to astronomy, physics and other sciences, but also because of the relation it has had and still has to such varied fields as art, religion, philosophy and the crafts” [Struik 1980, 26]. Aside from Struik’s general arguments, there exist important additional considerations, which make the study of the history of mathematics in Africa south of the Sahara even more indispensable.

Most histories of mathematics devote only a few pages to Ancient Egypt and to northern Africa during the ‘Middle Ages.’ Generally they ignore the history of mathematics in Sub-Saharan Africa and give the impression that this history either did not exist or, at least, is not knowable / traceable, or, stronger still, that there was no mathematics at all south of the Sahara (cf. the critics of [Lumpkin 1983; Njock 1985]). “Even the Africanity of Egyptian mathematics is often denied” [Shirley 1986b, 2]. Prejudice and narrow conceptions of both ‘history’ (cf. e.g. [Ki-Zerbo 1980]) and of ‘mathematics’ form the basis of such (Eurocentric) views (cf. e.g. [Joseph 1987, 1991]).

African countries face the problem of low ‘levels of attainment’ in mathematics education. “Math anxiety” is widespread. Many children (and teachers too!) experience mathematics as a rather strange and useless subject, imported from outside Africa. One of the causes thereof is that the goals, contents and methods of mathematics education are not sufficiently adapted to the cultures and needs of the African peoples (cf. e.g. [Eshiwani 1979, 346; Eshiwani 1983; Jacobson 1984]). Today’s existing African educational system is “unadapted and elitist” and “favours foreign consumption without generating a culture that is both compatible with the original
civilization and truly promising” [Ki-Zerbo 1990, 4]. The mathematical heritage of the peoples of Africa has to be valued and African mathematical traditions should be embedded into the curriculum [Ale 1989; Doumbia 1984, 1989b; Gerdes 1985a, 1986a, b, 1988d, 1990c; Langdon 1989, 1990; Mmari 1978; Njock 1985; Shirley 1986a, b]. This scientific legacy of Africa south of the Sahara is little known, and therefore research in this area constitutes a challenge to which an urgent response is necessary [Njock 1985, 4]. In addition, African-Americans and minorities of African descent all over the world feel the need to know their cultural-mathematical heritage [Campbell 1977; Frankenstein & Powell 1989; Zaslavsky 1973, etc.; Ratteray 1991]. More generally, both in highly industrialised and in Third World countries it is becoming more and more recognised that it is necessary to multi-culturalise the mathematics curriculum in order to improve its quality, to augment the cultural confidence of all pupils and to combat racial and cultural prejudice [D’Ambrosio 1985a; Ascher 1984; Bishop 1988a, b; Joseph 1987; Mellin-Olsen 1986; NCTM 1984; Zaslavsky 1989a, 1991].

Zaslavsky’s Africa Counts is a pioneering work on mathematics and its history south of the Sahara. She offers her book as “a preliminary survey of a vast field awaiting investigation” [Zaslavsky 1973a, vi]. Her task was not an easy one: in face of “the inadequacy of easily accessible material... “, she had to search “the literature of many disciplines — history, economics, ethnology, anthropology, archaeology, linguistics, art and oral tradition — ...” [Zaslavsky 1973a, vi]. Zaslavsky’s study deals with, what she calls, the sociomathematics of Africa: she considers “the applications of mathematics in the lives of African people, and, conversely, the influence that African institutions had upon the evolution of mathematics” [Zaslavsky 1973a, 7]. The concept of sociomathematics may be considered a forerunner of the concept of ethnomathematics. It is ethnomathematics as a

1 On mathematics education and the selection of élites, see [El-Tom 1984, 3].
Chapter 11

2 discipline that studies mathematics (and mathematical education) as embedded in their cultural context. For the (possible) relationships between ethnomathematics and the history of mathematics, see [D’Ambrosio 1985b] and (in the case of Africa) [Shirley 1986b] and [Gerdes 1990e]. The application of historical and ethnomathematical research methods has contributed, as will be shown, to a better knowledge and understanding of mathematics in the history of in Sub-Saharan Africa, as well as to an awareness of further mathematical elements in African traditions.

THE BEGINNINGS

As early evidence for (proto-) mathematical activity in Africa, Zaslavsky presented a bone dated at 9000-6500 B.C., unearthed at Ishango (Zaire). The bone has what appear to be tallying marks on it, notches carved in groups. Its discoverer, De Heinzelin, interpreted the patterns of notches as an “arithmetical game of some sort, devised by a people who had a number system based on 10 as well as a knowledge of duplication and of prime numbers” [Heinzelin, 1962, 110]. Marshack [1972], however, explained the bone as early lunar phase count. Their views, summarized in [Zaslavsky, 1973a, 17-19], were reproduced in [Fauvel & Gray, 1987, 5-7]. Later, Marshack re-evaluated the dating of the Ishango bone, setting it back from about 8000 B.C. to 20,000 B.C. [Marshack, 1991, 32]. Zaslavsky raises the question “who but a woman keeping track of her cycles would need a lunar calendar?” and concludes that “women were undoubtedly the first mathematicians!” [Zaslavsky 1991b, 4].

Bogoshi, Naidoo & Webb reported in 1987 on a still much older “mathematical artefact”: “A small piece of the fibula of a baboon, marked with 29 clearly defined notches, may rank as the oldest mathematical artefact known. Discovered in the early 1970s during an

2 In this sense “ethnomathematics” is closely related to the “sociology of mathematics” as founded by Struik [1942, 1986]. Cf. [Gerdes, 1993a] where an analysis of several forerunners of the concept of ethnomathematics and related concepts are discussed.
excavation of Border Cave in the Lebombo Mountains between South Africa and Swaziland, the bone has been dated to approximately 35000 B.C.” [Bogoshi, Naidoo & Webb 1987, 294]. They note that the bone “resembles calendar sticks still in use today by Bushmen clans in Namibia” [Bogoshi, Naidoo & Webb 1987, 294].

A research project looking for numerical representations in San (Bushmen) rock art has recently been started by Martinson [1992a, b]. From the surviving San hunters in Botswana, Lea and her students at the University of Botswana have collected information. Her papers describe counting, measurement, time reckoning, classification, tracking and some mathematical ideas in San technology and craft. The San developed very good visual discrimination and visual memory as needed for survival in the harsh environment of the Kalahari desert [Lea 1987, 1989, 1990a, 1990b, 1990c; Stott & Lea, 1993].

NUMERATION SYSTEMS AND NUMBER SYMBOLISM


During recent years, a whole series of research projects on spoken and written numeration systems in Africa are being carried out.

3 Several rock drawings and engravings of southern Africa are also interesting from a geometrical point of view. See for example [Dowson 1992].

These include studies on:

* counting in traditional Ibibio and Efik societies (I. Enukoha, University of Calabar, Calabar, Nigeria);
* numeration among the Fulbe (Fulani) (S. Ale, Ahmadu-Bello-University, Bauchi, Nigeria; cf. [Ale 1989]);
* pre-Islamic ways of counting (Y. Bello, Bayero University, Nigeria);
* mental arithmetic, algorithms and counting among the different ethnic groups of Nigeria (Ahmadu-Bello-University, Zaria);
* pre-colonial numeration systems in Burundi (J. Navez, University of Burundi, Bujumbara);
* learning of counting in Côte d’Ivoire [Tro 1980; Zepp 1983c];
* numeration systems used by the principal linguistic groups in Guinea (S. Oulare, University of Conakry);
* counting among the various ethnic groups in Kenya (J. Mutio, Kenyatta University, Nairobi);
* oral and possible graphic numeration systems from Zaire [Mubumbila 1988];
* numeration and geometric figures in Great Zimbabwe [Mubumbila 1992];
* traditional counting in Botswana (H. Lea, University of Botswana, Gaborone);
* number and pattern in selected cultures in Uganda (E. Segujja-Munagisa).

An important study – from the point of view of its contents and

the methodological debate it initiates – is E. Kane’s doctoral dissertation [Kane 1987] on “The spoken numeration systems of West-Atlantic groups and of the Mandé.” Kane (Cheik-Anta-Diop-University, Dakar-Fann, Senegal) analyses numeration in about twenty languages spoken in Senegal. He realised the necessity of basing his research on ethnomathematics, trying to understand mathematical ideas in relationship to the general culture in which they are embedded. Therefore he did preparatory research in four domains: African linguistics, history of numeration systems, works of ethnologists and African languages spoken in Senegal (as understood by interviewing many speakers of the same and different languages). He shows that oral numeration systems, like the one of the Mandé, are susceptible to reform and evolution. Kane develops a methodology for the analysis of numeration systems that is adapted to the specificities of ‘oral cultures’.


Zaslavsky dedicated a chapter to number symbolism and taboos on counting [Zaslavsky 1973, 52-57]. The examples she gave are based, among others, on a study of number symbolism among the Ijo in Nigeria [Williamson & Timitimi 1970]. For example, the number three is associated with men, the number four with women. Ojoade [1988] published a paper on the number 3 in African lore, highlighting the sacredness, mysticism and taboos attached to it (cf. also [Nicolas 1968]). In [Page 1987] objects of African art, mostly from the Yoruba (Nigeria) are analysed as functions of the involved repetitions. The twofold objects evoke the standard dichotomies: good/bad, life/death; the threefold objects sometimes evoke a hierarchy; the fourfold objects may be associated with the directions in space. By systematically searching the ethnographical literature as well as novels, biographies,
etc. it is likely that a good deal more information on number symbolism in African cultures will be found (see also [Alberich 1990]). For instance, the anthropological study of Thornton explains the significance of the number 9 among the Iraqw of Tanzania [Thornton 1980, 96, 167, 183]. Number symbolism may have a rational basis. E.g. Makhuwa basket makers in northern Mozambique call odd numbers or odd quantities of plant strips ‘ugly,’ and they have good reasons for doing so as shown in [Ismael 1991]. For an earlier discussion of ‘even’ and ‘odd’ numbers in basketry see [Gerdes 1985a]. Makhuwa hunters are always acting in groups of an even number [Ismael 1991]. [Mubumbila 1988] discusses the symbolic expression of numbers in Luba cosmogony (Zaire), for example, the significance of ‘even’ and ‘odd’ and the use of ‘numbers of peace’ like 4, 12, 24, 48, and 96. Certainly, further collecting of oral data may throw new light on African numerology.

GAMES, RIDDLES AND PUZZLES

games of alignment [Shisima (Kenya), Achi (Ghana), Murabaraba (Lesotho)], ‘struggle-for-territory’ games (Sega (Egypt), Kei (Sierra Leone), and ‘Mancala’ games, both two-row versions [Oware (Ghana), Awélé (Ivory Coast), Ayo and Okwe (Nigeria)] and four-row versions [Omweso (Uganda), Tshisolo (Zaire)]. Bell and Cornelius (1988) give some information on Achi (Ghana), Dara (Nigeria), Sega (Egypt) and on ‘Mancala’ games. [Retschitzki 1988] and [N’Guessan 1988] analyse the learning of strategies and tactics of the ‘awélé’ game. The important research of Doumbia and her colleagues at the Mathematical Research Institute of Abidjan (Côte d’Ivoire) has focused on traditional African games. It deals with classification and solution of mathematical problems posed by the games, and explores the possibilities of using these games in the mathematics classroom. Their studies [Doumbia 1989b; Doumbia & Pil 1992; Doumbia & N’guessan 1994] reveal that the rules of some games, like Nigbé Alladian, indicate an empirical knowledge of probabilities, a finding that will certainly stimulate further research. Vergani (Open University, Lisbon) is preparing a monograph on the mathematical aspects of intellectual games in Angola. Mve Ondo (Omar-Bongo-University, Gabon) published a study on two ‘calculation games’, i.e. the ‘Mancala’ games, Owani (Congo) and Songa (Cameroon, Gabon, Equatorial Guinea) [Mve Ondo 1990]. The possible relationship between visual memory and concentration as necessary factors for success in many African games (cf. [Paul 1971]) and the development of mathematical ideas also deserves further attention.

Zaslavsky [1973, 109-110] presents a riddle from the Kpelle (Liberia) about a man who has a leopard, a goat, and a pile of cassava leaves to be transported across a river, whereby certain conditions have to be satisfied: the boat can carry no more than one at a time, besides the man himself; the goat cannot be left alone with the leopard, and the goat will eat the cassava leaves if it is not guarded. How can he take them across the river? [Ascher 1990] places this river-crossing problem in a cross-cultural perspective and analyses mathematical-logical aspects of story puzzles of this type from Algeria, Cape Verde Islands, Ethiopia, Liberia, Tanzania and Zambia. More difficult to solve is an ‘arithmetical puzzle’ from the Valuchazi (East Angola and Northwest Zambia), recorded and analysed by Kubik: “This..dilemma tale is about three women and three men who want to cross a river in
order to attend a dance on the other side. With the river between them there is a boat with the capacity for taking only two people at one time. However, each of the men wishes to marry all the three women himself alone. Regarding the crossing, they would like to cross in pairs, each man with his female partner, but failing that any of the other men could claim all the women for himself. How are they crossing?” [Kubik 1990, 62]. In order to solve the problem or to explain the solution, auxiliary drawings are made in the sand. [Béart 1955, 744-745] presents a series of riddles from West Africa. [Fataki 1991] describes riddles he learnt as a child in Uganda.

A ‘topological’ puzzle from the Bambala (Bakuba, Congo) is recorded in [Torday & Joyce 1911, 90, 96]. It consists of two pieces of calabash and a string arranged as in Figure 1. The player has to separate one of the calabash pieces from the string without cutting or untwisting the string.

![Bambala ‘topological’ puzzle](image)

Bambala ‘topological’ puzzle
(Drawn after Torday & Hoyce 1911, 90)

Figure 1

A more complicated puzzle from the Guinean forest is the game called *pèn* [Béart 1955, 413]. It uses an instrument (see Figure 2) produced in the following way. A stick is perforated in the middle. Through the hole passes a loop of a string and then both ends of the string pass through the loop. Perforated nuts are strung on the thread and thereafter the endpoints of the string are tied to the endpoints of the sticks. Now the player has to displace the nuts from the big loop on the left side to the big loop on the right side without cutting or untwisting
the string. Other examples of ‘topological’ puzzles from West Africa are recorded in [Béart 1955, 413-419].

![West African game ‘pèn’](image)

**(Figure 2)**

**GEOMETRY**

*Art and symmetries*

Njock (University of Yaoundé, Cameroon) characterises the relationship between African art and mathematics as follows: “Pure mathematics is the art of creating and imagining. In this sense black art is mathematics” [Njock 1985, 8].

Mathematicians have mostly been drawn to the analysis of symmetries in African art. Symmetries of repeated patterns may be classified on the basis of the 24 different possible types of patterns, which can be used to cover a plane surface (the so-called 24 plane groups due to Federov). Of these, seven admit translations in only one direction and are called strip patterns. The remaining 17, which admit two independent translations, are called plane patterns. In chapter 14 of *Africa Counts*, Crowe applies this classification to decorative patterns that appear on the raffia pile cloths of the Bakuba (Congo) (see also [Crowe 1971]), on Benin bronzes, and on Yoruba adire cloths (Nigeria), showing that all seven strip-patterns occur and many of the plane patterns. Crowe continued this research and published a
catalogue of Benin patterns [Crowe 1975] and a symmetry analysis of the smoking pipes of Begho (Ghana) ([Crowe 1982a]; cf. also [Crowe 1982b]). In Washburn and Crowe [1988] a number of patterns from African contexts are classified in the same way. The use of the crystallographic groups in the analysis of symmetries in African art underscores and attests to the creative imagination of the artists and artisans involved and their capacity for abstraction (cf. [Meurant 1987]). More recently [Washburn 1990] showed how a symmetry analysis of the raffia patterns can differentiate patterns produced by the different Bakuba groups. She also takes into account the way in which the artists and artisans themselves classified and analysed their symmetries. In [Gerdes & Bulafo 1994] it is explained which calculations are done by basket weavers from Inhambane Province in Mozambique to produce strip patterns on their ‘sipatsi’ handbags. Their book also presents a catalogue of more than one hundred patterns (see the example in Figure 3). The creativity of the ‘sipatsi’ weavers – formerly only women – also expresses itself in the fact that they invented strip patterns belonging to all of the seven different, theoretically possible, symmetry groups (see also [Gerdes 1994d; Uaila, 1992]).

Why do symmetries appear in human culture in general, and in African craftwork and art, in particular? This question is addressed by Gerdes in a series of studies that analyse the origin of axial, double axial, and rotational symmetry of order 4 in African basketry (see [Gerdes, 1985a, 1987, 1989a, 1990c, 1991c, 1992b]). In [Gerdes 1991b] it is shown how fivefold symmetry emerged quite “naturally”
when artisans were solving problems in basket weaving. The examples chosen from Mozambican cultures range from the weaving of handbags, hats, and baskets to the fabrication of brooms.

Zaslavsky [1979] gives some examples of strip and plane patterns, and of bilateral and rotational symmetries, occurring in African art, architecture and design. [Langdon 1989, 1990] describe the symmetries of ‘adinkra’ cloths (Ghana) and explores possibilities for using them in the classroom. In a similar perspective, [Harris 1988] describes and explores not only the printing designs on plain woven cloths from Ghana, but also symmetries on baskets from Botswana and ‘buba’ blouses from the Yoruba (Nigeria). [Gerdes 1992c] presents a series of traditional African designs with fourfold symmetry and suggests ways of using them in a didactical context to discover / reinvent the Pythagorean theorem.

Example of a monolinear ‘(lu)sona’ – ‘sand drawing’
(Drawn after Fontinha 1983, 183; Pearson 1977, 144)

Figure 4
Networks, graphs or ‘sand-drawings’

One section of *Africa Counts* [Zaslavsky 1973a, 105-109] was devoted to networks based on Torday’s information on the (Bu)Shongo (Congo) and Bastin’s study of decorative art of the Tchokwe (Angola) [Torday 1925; Bastin 1961]. At the time she did not have access to the ethnographical information on such networks published in [Baumann 1935, 222-223], [Hamelberger 1952] and [Santos 1961]. Since the publication of *Africa Counts*, large ethnographical collections of networks have become available: [Pearson 1977] deals with ‘sand-graphs’ observed in the 1920s in the Kwandu-Kuvanga and Muxiku provinces of Angola; [Fontinha 1983] considers ‘sona’ or ‘sand-drawings’ collected principally among the Tchokwe of north-eastern Angola during the 1940s and 1950s; and [Kubik 1986, 1987a, 1987b, 1988] report on networks observed among the (Va)luchazi in north-western Zambia during the 1970s. 6 In order to facilitate the memorisation of their standardised ‘sona’ (see the example in Figure 4), the drawing experts used the following mnemonic device. After cleaning and smoothing the ground, they first set out an orthogonal net of equidistant points with their fingertips. Next one or more lines are drawn that ‘embrace’ the points of the reference frame. By applying their method the drawing experts reduce the memorisation of a whole drawing to that of mostly two numbers (the dimensions of the reference frame) and a geometric algorithm (the rule of how to draw the embracing line(s)). Most drawings belong to a long tradition (cf. [Redinha 1948]). They refer to proverbs, fables, games, riddles, animals, etc. and play an important role in the transmission of knowledge and wisdom from one generation to the next. In Kubik’s view the ‘sona’ “transmit empirical mathematical knowledge” [Kubik 1987a, 450]. The ‘sona’ geometry is a “non-Euclidean geometry”: “The forefathers of the Eastern Angolan peoples discovered higher mathematics and a non-Euclidian geometry on an

6 Dirk Struik [1948] was probably the first historian of mathematics to call attention to the geometry of this type of figure referring to sand drawings from the Malekula Islands (Oceania).
empirical basis applying their insights to the invention of these unique configurations” [Kubik 1987b, 108]. Kubik calls attention to the symmetry of many ‘sona’, the implicit rules for construction and rules for anchoring figures of the same type. The ethnographical publications of collections of ‘sona’ have attracted the attention of mathematicians. Ascher and Gerdes conducted research on the ‘sona’, independently of one another. [Ascher 1988, 1991] deal with geometrical and topological aspects of ‘sona’, in particular with symmetries, extension, enlargement through repetition, and isomorphy. [Gerdes 1989a, 120-189] and [Gerdes 1993b, vol.1] analyse symmetry and monolinearity (i.e. a whole figure is made up of only one line) as cultural values, classes of ‘sona’ and corresponding geometrical algorithms for their construction, systematic construction of monolinear ground patterns, chain and elimination rules for the construction of monolinear ‘sona’. It is suggested that the ‘drawing experts’ who invented these rules probably knew why they are valid, i.e. they could prove in one or another way the truth of the theorems that these rules express. He advances also with the reconstruction of lost symmetries and monolinearities by means of an analysis of possible drawing errors in reported ‘sona’ (for an introduction to his research findings, see [Gerdes 1990d, 1991d, 1991e]. Inspired by his historical research findings, Gerdes experimented with the possibilities of using the ‘sona’ in mathematics education, in order to value and revive a rich scientific tradition that had been vanishing (see [Gerdes 1988a and b; 1989a, b, and c; 1990a; 1991a and f, 1993b (vol.2) and d]; cf. [Ratteray 1991]). He also initiated a mathematical exploration of the properties of some (extended) classes of ‘sona’ (see [Gerdes 1989a, 288-297] and [Gerdes 1993b, vol.2]). In a similar way, Kubik’s research stimulated a mathematical investigation by Jaritz on a particular class of ‘sona’ [Jaritz 1983].

Monolinear patterns appear also in other African contexts. For instance, [Prussin 1986, 90] displays a symmetric monolinear pattern on a Fulbe warrior’s tunic from Senegal (see Figure 5). The study of

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the types and spread of monolinear patterns throughout the African continent deserves further research (cf. [Gerdes 1994a]).

![Design on a warrior’s tunic of the Fulbe (Senegal)](Drawn after Prussin 1986, 90)

Figure 5

**Geometry and architecture**

Chapter 13 of *Africa Counts* (see also [Zaslavsky 1989b]) is dedicated to geometric form in architecture. More information on the geometric shapes and on the ornamentation of traditional African buildings may be found in [Denyer 1978]. [Anon. 1987] presents a bibliography on African architecture. [Mubumbila & Bum 1992] booklet on African architecture gives particular attention to shape and geometric form. Prussin has called attention to the fact that in West Africa the mathematician-scholar and the architectural designer-builder might often be the same person [Prussin 1986, 208]. She refers to the relationship between magic squares and the structure of domes, and remarks that “a number of ‘adinkra’ [Ashanti, Ghana] stamp patterns, directly associated with Islam, were also used in the architectural setting” [Prussin 1986, 240]. For more information on the use of magic squares in West Africa, see [Béart 1955, 746-748].
[Rohrman 1974], [Matthews 1974], [Courtney-Clarke 1986] and [Changuion et al. 1989] describe house decoration and mural painting in Southern Africa, in particular among the Ndebele. A publication of [NTTC 1976] gives a catalogue of geometric patterns used on house walls in Lesotho (see the examples in Figure 6).

Examples of ‘litema’ patterns (Lesotho)
(Drawn after NTTC 1976)
Figure 6

Also in West Africa according to [Courtney-Clarke 1990] are it mostly the women who decorate the walls of their houses with geometrical figures. Each year after the harvest, they gather to restore and paint their mud dwellings which have been washed clean by the rains of the wet season. These studies may serve as a starting point for further research on geometry and ornamentation of buildings. [Eglash & Broadwell 1989] and [Eglash 1994] are interested in possible relationships between modern fractal geometry and traditional settlement patterns in Africa. [Gerdes 1985a, 1990c, 1992b] describe the geometrical know-how used in laying out circular or rectangular house plans in Mozambique.

Uncovering ‘hidden’ geometrical ideas

Many ‘mathematical’ ideas and activities in African cultures are
not explicitly mathematical. They are often intertwined with art, craft, riddles, games, graphic systems, and other traditions. The mathematics is often ‘hidden’ or ‘implicit’ (see [Zaslavsky 1994]). How may this ‘hidden’ knowledge be uncovered? As some traditions are nowadays (becoming) obsolete, this ‘uncovering’ often requires a tentative reconstruction of knowledge, as it existed in the past. [Gerdes 1985a, 1990c, 1992b] explores the concept of ‘hidden’ mathematics and develops some methods of ‘uncovering’ and reconstructing ‘hidden’ geometrical thinking. One of these methods may be characterised as follows: when analysing the geometrical forms of traditional objects – like baskets, mats, pots, houses, fish traps – the researcher poses the question: why do these material products possess the form they have? In order to answer this question, the researcher learns the usual production techniques and tries, at each stage of the production process, to vary the forms. Doing this, the researcher observes that the form generally represents certain practical advantages and that frequently it is the optimal or only possible solution of a production problem. By applying this method, it becomes possible to bring to the fore knowledge about the properties and relations of circles, angles, rectangles, squares, regular pentagons and hexagons, cones, pyramids, cylinders, symmetry, etc., that were probably involved in the invention of the production techniques under consideration.

One of the many areas of African culture, which have not been studied as yet in the light of their inherent mathematical aspects, is that of string figures. Elsewhere such an analysis has already begun: [Paula & Paula 1988] studied the geometry of string figures among the Tapirapé Indians in Brazil; [Moore 1986] analysed string figures from the Navaho and other North American Indians and explored their potential for mathematics education. In the case of Africa south of the Sahara, the studies on string figures of Angola [Leakey & Leakey 1949], Botswana [Wedgwood & Schapera 1939], Central Africa [Cunnington 1906], Ghana [Griffith 1925], Liberia [Hornell 1930], Nigeria (Yoruba) [Parkinson 1906], Sierra Leone [Hornell 1928, 1930], Southern Africa [Haddon 1906], Sudan [Hornell 1940, Evans-Pritchard 1955], Tanzania (Zanzibar) [Hornell 1930], West Africa [Béart 1955, 398-412] and [Lindblom 1930] may serve as a starting point.

In general, once the mathematical character or aspects of cultural
elements are recognized, one may try to track the history of the mathematical thinking involved and its (possible) relationships to other cultural-mathematical ‘threads’ and try to explore their educational and scientific potential.

STUDIES IN RELATED DISCIPLINES

As the history of mathematics in Africa should not be considered in isolation from the development of culture in general, or be dissociated from the evolution of art, cosmology, philosophy, natural sciences, medicine, graphic systems and technology in particular, the historiography of African mathematics has to take into account the research findings from other disciplines. The following overviews cover some of the pertinent literature.

[Pappademos 1983] presents an outline of Africa’s role in the history of physics. [Weule 1921] studied early forms of mechanics, based on his fieldwork in eastern Africa. [Lynch & Robbins 1983] analyse evidence from Namoratunga, a megalithic site in North Western Kenya, that suggests that a prehistoric calendar based on detailed astronomical knowledge was in use in eastern Africa (c.300 B.C.). [Dundas 1926] describes pre-colonial time-reckoning among the Wachagga (Kilamanjaro-region): the year is divided into twelve months; each month has thirty days and is divided into six periods of five days each. The ‘topology of time’ among the Iraqw of Tanzania is analysed by [Thorton 1980]. Concepts of time, time-reckoning and cosmology of the Kagura (eastern Africa), of the Dwala (Cameroon) and of the Tiv (Nigeria) have been described and analysed by [Beideman 1963] and [Bohannan 1953] (cf. also [Booth 1975]; [Kagame 1976]). [Lacroix 1972] discusses the time expressions in some West African languages. Traditional African calendars constitute one of the research themes of the ‘Thought Systems in Black Africa’ study group in Paris [Baker 1987, 53]. [Obenga 1987] reviews the literature on astronomical knowledge in Ancient Egypt, among the Borana (Ethiopia), Dogon (see also [Griaule & Dieterlen 1950; Zahan 1951; Griaule 1951, 9-13], Lobi, Bambara (West Africa), Vili (Congo), Fang (Cameroon, Equatorial Guinee, Gabon), and Mbochi (Congo, cf. [Obenga 1982]). Keller’s study [1902] deals with the astronomical views of the Isubu in Cameroon. [Adams III 1983a, b;
Brul 1932; Junod 1974, vol.2, 268-274; Legesse 1973; Vergiat 1937] may also be consulted on astronomy and on calendars in Africa.

An early attempt to analyse craftwork and technology in eastern Africa in a historical perspective is Stuhlmann’s study [1910]. Studies of this type are, however, relatively rare. As Thomas-Enegwali stresses, despite the significance of the history of technology there is a relative dearth of writings on the issue in African historiography in general and in Nigerian historiography in particular. Her evaluation of the role of oral historiography in the reconstruction of the history of technology is important from a methodological point of view. One of the major problems [Thomas-Enegwali 1988, 69] with which the historian of technology is confronted in the course of fieldwork is the understandable reluctance of practitioners to divulge technological secrets so that they can maintain some measure of competitive power over their potential and actual rivals. Today technicians and craftsmen are often unaware of the precise scientific and engineering principles when they use ancient techniques in the making of material objects (iron technology, textile production, basketry, woodwork). Therefore, according to Thomas-Enegwali, the researcher “must be able to identify the underlying principles at play in the process” [Thomas-Enegwali 1988, 70]. Studies on traditional medicine, biologically-based warfare, the control of water-based diseases, glassmaking technology and metallurgy in Nigeria, and on gold-mining in pre-colonial Zimbabwe and diamond-mining in Sierra Leone are included in [Thomas-Enegwali 1992b]. [Thomas-Enegwali 1992a, 1993] include studies on traditional medicine, religion and science, textile technologies, metal technology, mechanics, military technology and engineering, microbiology and traditional food processing, and gender and technology in Nigeria.


AFRICA SOUTH OF THE SAHARA, NORTH AFRICA AND THE OUTSIDE WORLD

The relationships between the development of mathematics in Sub-Saharan Africa and the development of mathematics in Ancient Egypt, in both Hellenistic and Islamic northern Africa, and across the Indian and Atlantic oceans, also deserve further study.

In connection with ancient Egypt, many open questions still exist. For example, do relationships exist between the duplication and symmetry patterns in pre-Bantu rock paintings in Mozambique, the ‘binary combinatorics’ of Pedi augurs in Transvaal (South Africa, cf. [Junod 1974, 559-564]), elements of a binary structure in some African 202

On the basis of mathematical ideas involved in the invention and use of basketry techniques (e.g. woven pyramidal funnels in Mozambique and Zaire) and mat making (e.g. circular ‘spiral’ mats), and taking into account the cultural linkages between ancient Egypt and the rest of ‘Black Africa’ [cf. Diop 1981], it is possible to formulate new hypotheses as to how the ancient Egyptian formulas for the area of a circle [Gerdes 1985b] and for the volume of a truncated pyramid [Gerdes 1985a, 1990c] may have been found.

Throughout history there have been many and varied contacts between Sub-Saharan Africa and North Africa. Kani’s paper “Arithmetic in the pre-Colonial Central Sudan” [Kani 1992a] considers *Ilm al-Hisab* (arithmetic) as part of the Islamic sciences introduced some time after the 11th century in Nigeria, first in Kanem-Borno and later, probably 15th century in Hausaland. Arithmetic being taught in both ‘secular’ and *Islamiyya* schools, was used in the courts (calculation of inheritance), collecting and distributing *zakat* (poor-dues), business and land surveying. Scholars of Hausaland and Borno consulted Coptic solar calendars in determining their economic activities, especially agricultural ones. Kani concludes his paper with the following remarks: “Despite the availability of a great deal of literature on medicine, astrology, arithmetic and other related sciences, written in Arabic, Fulfulde, Hausa and other languages, little effort has been made to systematically study these sciences within the historical perspective. The intellectual output of the *Ulama* (scholars) in this area has been wrongly classified by our contemporary historians and social scientists under the rubric of ‘mysticism’. A serious investigation into the literary output of the scholars of western and central Sudan, however, may reveal the fact that these scholars had explored agricultural, medicinal, astronomical and mathematical
sciences long before the advent of colonial rule” [Kani 1992a, 38] (see also [Kani 1992b]). [Zaslavsky 1973a, 138-151] discusses the work of Muhammed ibn Muhammed from Katsina (now northern Nigeria) on chronograms and magic squares. Muhammed ibn Muhammed, who had been a pupil of Muhammed Alwali of Bagirmi, made a pilgrimage to Mecca in 1730 before he died in Cairo in 1741. [Kani 1986] discussed the work of the same Muhammed ibn Muhammed al Katsinawi on magic squares and numerological patterns. In 1990 a manuscript of his was found in Marrakesh (Morocco) [Djebbar, personal communication]. [Prussin 1986, 76, 147] refers to the use of magic squares in amulets among the Fulbe, and in Niger, Benin and Timbuktu (Mali). [Thomas-Emeagwali 1987] reflects on the development of science in the Islamic world and its diffusion into Nigeria before 1903. [Diop 1960, 167] refers to the study of formal logic in Timbuktu and Lapousterle (Bamako, Mali) is preparing a study on the contents of three mathematical manuscripts, written in Arabic, that belong to the Ahmad Baba Library in Timbuktu. One of the three manuscripts, whose calligraphy is typical of Sub-Saharan Africa, seems to have been written by a mathematician from Mali, al-Arwani. The other two contain references to medieval mathematicians from the Maghreb. Systematic search in libraries and archives will probably lead to the discovery of more mathematical manuscripts from Muslim scholars south of the Sahara. Other sources written in Arabic, on mathematical activities south of the Sahara may also exist. An example from the eastern coast of Africa is provided by the comments of traveller Ahmad Ibn-Madjid at the end of the 15th century, on the counting of the Wac-Wac, probably not a Bantu but a Khoi-San people, living at that time in southern and central Mozambique (see [Chumovsky & Jirmounsky 1957]).

The possible mutual links between the development of mathematics across the Indian Ocean still remain to be studied (for general analysis of the historical relations across the Indian Ocean, see [UNESCO 1980]). What about the possible influence of mathematical ideas from slaves from the African continent and from India and Indonesia on the development of mathematics in Madagascar, Mauritius and other islands?
SLAVE TRADE AND COLONISATION

What kind of mathematics was brought to the Americas by the slaves? Which mathematical ideas have survived in one way or another? ‘Mancala’ and perhaps other games with mathematical ‘ingredients’ are played in the Caribbean and may be compared with their ‘ancestors’ in Africa. [Ferreira 1982, 2] refers to a study by one of his students on geometric symbology in ‘Ubanda’ and ‘Candomblé’ in Brazil. This research area remains almost virgin.

In Fauvel & Gerdes’ paper on Thomas Fuller (1710-1790), the African slave and calculating prodigy, shipped to America in 1724, it is suggested that ethno-mathematical research may complement the analysis of written sources [Fauvel & Gerdes 1990]. Fuller’s exceptional abilities can be understood only through closer examination of the cultural context that stimulated their development. Like the professional knowledge of the Tchokwe smiths, the knowledge of the Tchokwe drawing experts (akwa kuta sona) was mostly secret. When a blacksmith or a drawing expert was taken prisoner and sold as a slave, his specific professional knowledge could easily disappear completely from his village or region. Thus the slave trade was extremely destructive to the development of the existing mathematical traditions and potential, since it broke down the professional continuity and deprived Africa of its bearers of mathematical knowledge and skill, such as Thomas Fuller. Recent ethnomathematical research in Nigeria, as summarized by [Shirley 1988], shows the survival, nonetheless, of a rich tradition of mental calculations among illiterate people. It would be interesting and valuable to search for new records with data on the geographical or ethnic origin of Thomas Fuller, and to correlate this information with ethnomathematical and historical research on the same region or ethnic group.

One possible way in which the slave trade influenced the development of arithmetical knowledge in Africa has been described as follows by Clarkson in 1788: “Perhaps brought to the front or produced by the necessity of competing with English traders armed with pencil and paper, many of the old-time slave-dealers of Africa seem to have been ready reckoners, and that, too, for a practical purpose ... The ship captains are said to have complained that it
became more and more difficult to make good bargains with such sharp arithmeticians” (cited in [Scripture 1891, 2]). It would be interesting to explore further this and other possible influences, such as the disappearance or undermining of traditional African mathematical education by the physical elimination or ‘exportation’ of the bearers of mathematical knowledge.

It might be suggested that it would be interesting to search African literature, including autobiographies (cf. the autobiographical essays [Fataki 1991; Mugambi 1991]), for information on mathematics education in the colonial period and the reaction to it (e.g. attitudes towards mathematics and mathematics teachers). A systematic analysis of the ideas (and prejudices) of missionaries, colonial administrators and educators about the mathematical capacities of their African ‘subjects’ (e.g. the comments of Junod on the ‘lack of mathematical ability’ in Mozambique (Junod, 1935, 151-155, 576-577) and of their implications, might be worthwhile and revealing for the present and future generations.

POST-INDEPENDENCE


An interesting theme in the recent history of mathematics education in Africa, which seems to deserve study, is that of the emergence and evolution of continental and regional mathematics curriculum development projects, like the African Mathematics Program (for a short historical overview see [Wilson 1981, 195-199]), School Mathematics Project for East Africa, Joint Mathematics
Project, East African Regional Mathematics Program, and West African Regional Mathematics Program. The history of mathematical associations and journals, of mathematical departments and schools, are other possible research topics.

The introduction and spread, in Sub-Saharan Africa, of new mathematical research areas and related fields such as statistics, informatics, and computer science in relation to the introduction and spread of new technologies, like the use of computers, have not been studied as yet, nor have their implications for the countries involved.

ACKNOWLEDGEMENTS

The author thanks Salimata Doumbia (Côte d’Ivoire), Claudia Zaslavsky (USA) and the editor of Historia Mathematica for their comments on an earlier version of this paper.

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Chapter 11


Appendices
Chapter 12

Conditions and Strategies for Emancipatory Mathematics Education in Underdeveloped Countries *

Reflecting on the author’s experiences in Mozambique, he suggests some conditions and strategies for mathematics education to become emancipatory in underdeveloped countries.

Point of departure: mathematics education cannot be neutral

By its physical and intellectual labour, mankind is able to create an ever more human society. Reflecting on its realisations, discovering the laws of nature and society, mankind creates its material and intellectual tools to transform reality, both nature and society. Mathematics constitutes an integrated body of those means to understand and transform reality. An ever more human society is the most rational direction. However, mankind has at its disposal, nowadays, the means to destroy itself. Neither mathematics, nor

mathematics education nor mathematicians can be indifferent towards these diametrically opposed possibilities:

<table>
<thead>
<tr>
<th>more human</th>
<th>more inhuman</th>
</tr>
</thead>
<tbody>
<tr>
<td>peaceful</td>
<td>war-like</td>
</tr>
<tr>
<td>liberating</td>
<td>oppressing</td>
</tr>
<tr>
<td>creating</td>
<td>destructive</td>
</tr>
<tr>
<td>emancipating</td>
<td>exploiting</td>
</tr>
</tbody>
</table>

The history of Mozambique shows very clearly that mathematics education cannot be neutral [see Gerdes 1980, 1981 b]. During the Portuguese domination, mathematics was taught, in the interest of colonial capitalism, only to a small minority of African children [see Mondlane 1969]. And those Mozambicans were taught mathematics to be able to calculate better the hut tax to be paid and the compulsory quota of cotton every family had to produce. They were taught mathematics to be more lucrative “boss-boys” in South African mines. After ten years of the people’s war, led by the liberation movement FRELIMO, Portuguese colonialism was defeated and Mozambique achieved its Independence in 1975. Post-independence objectives of mathematics education are in the service of the construction of a socialist society [cf. Machel 1977, Ganhão 1978]. Mathematics is taught to “serve the liberation and peaceful progress of the people”. Mathematics is taught to place its applications within the reach of the worker and peasant masses. Mathematics is taught to stimulate the broad masses to take an interest and delight in mathematical creation. These are general objectives. But how are they to be achieved?

Mathematics education for emancipation. How?

Independence and the option for socialism have implied an overall democratisation of mathematics education in Mozambique: the majority of children, and hundreds of thousands of adults, have already gained access to the science of mathematics; the mathematics teaching
professions has been opened up for children of peasants and labourers; and the discussion on how to improve the quality of mathematics education is not any more reserved for an elite but is becoming the object of reflection of the mass the teachers, from adult education and primary school teachers to university professors [cf. Gerdes 1981 b]. Overall democratisation is a necessary but not sufficient condition for mathematics education to become really emancipatory – everyone mastering mathematics and capable of thinking mathematically, to the benefit of society as a whole. On the basis of the author’s experience in the training of mathematics teachers, some other necessary conditions will be presented.

Problemising reality in classroom situations

Let us briefly listen to some dialogues occurring in our lessons:

* A student enters the classroom, late and noisy. Confrontation. Could your parents study in colonial times? No ... Who pays for your study? (With Independence, education became free of charge in Mozambique) The peasants and labourers ... How many days does a peasant have to work to pay for your being late? What do you mean? How much does it cost? What do we have to know to calculate it? (In a concrete case it was calculated, by the class, that one student coming one hour late corresponds to throwing away one day of a peasants’ work.)

* Newspaper reading, a photograph: a truck crossing a bridge over the Changana River, the bridge collapsed. Why? Was the truck too heavy? The bridge ill-built?

* In a report by the Centre of African Studies at our University it is told that a tractor driver drove his machine at full speed for two hours to fetch only three loaves of bread (from a state farm in Moamba). Why did he do so? Was it justifiable? Is it reasonable? Why? Why not? How can his behaviour be explained? How can the cost be evaluated?

* For a certain period, sugar production was going down. Why? Changes in the way of paying the labourers had been introduced:
from payment in terms of the number of rows of sugarcane cut down to the number of kilograms of sugarcane. Why? How can we explain the economic consequences? How can production be raised? Is mathematics involved?

* Our country suffered floods in 1977 (Limpopo River) and in 1978 (Zambeze River). Why didn’t we have any floods in the early eighties? Is mathematics involved? How?

* To combat speculation a food distribution system has been introduced in the city of Maputo (1981). How much rice for each person this month? How much rice a family of five persons can eat each day?

These dialogues are examples of problemising reality (Freire’s terminology). “Reality is thrown before our feet” (in the deep sense of the original Greek “προ-βαλλειν”); one cannot fly from reality, one has to reflect. Experience shows that problemising reality leads to consciousness, to awareness of the relevance of mathematics as a tool to understand and transform reality. It leads to political awareness (as in the case of the student who came late); to physical awareness (as in the case of the bridge that collapsed); to economical awareness (as in the example of the tractor driver), etc.

Let us give some more examples of problemising reality before drawing a second conclusion – examples that give a meaningful, rich context to mathematics education.

* In Cabo Delgado province some students were told that the area of a cotton field is so many metres. Area measured in m and not in m$^2$? Does it make sense? Why?

* Mozambique’s soap production was 16300 t in 1980 and 23700 t in 1981. What will the national needs be in 1990? How can they be calculated? What will the soap production be in 1990? Regular growth of production capacities or not? Regular in what sense? Why?

* In order to win the battle against underdevelopment during the decade of the eighties, the People’s Assembly approved a National State Plan for the decade. We need $x$ school buildings by 1990. In 1981 we already had $y$ school buildings and will
construct $z$ more school buildings. How do we have to extend production capacities for the construction of schools? Regular growth? How regular? Linear? Parabolic? Exponential? (A possible entrance to the study of geometric or exponential progressions.)

* Women fetch water in their tins, cylindrical cans, on their heads. Are they exploited? Development will lead to piped water in each house. And in the meantime? Could the tin cans be less heavy? Could more cans be produced out of the same quantity of raw materials? How? What changes are implied? (A possible entrance to the study of differential calculus.)

Here it should be emphasized that a problemising reality approach leads to a real understanding of reality, it leads to an understanding of mathematics as a tool to transform reality.

Let us give one more example to reinforce this second conclusion [see Gerdes 1982].

* Rickettsiosis is a disease that kills a lot of cattle in our country. Veterinarians discovered a medicine to cure sick animals (terramycin). How much medicine is needed for each animal? This quantity depends on what? On the colour of the animal? Its height? How does this quantity depend on the weight of the cow? Linear dependence? Exponential? Why? How do we weigh a cow? Only on well-equipped farms will you find scales able to do so. What can be done in the less developed countryside? The weight of an animal is related to ...? Why? How? Is it possible to determine the weight of a cow in an indirect way? A cow knows how to swim? What does this imply? Relation weight-volume? How can the volume of a cow be determined? Let us take a very close look at the cow: approximation to a cylinder! (See Figure 1). And so on.

Reality can be changed. More animals will survive. People will have more meat and milk. Improved calculations lead to a reduction in the quantity of medicine. And, as these medicines still have to be imported, well-applied arithmetic will save foreign currency.
Approximate cylinder

Figure 1
Creating confidence

For mathematics education to become emancipatory, it is necessary to stimulate confidence in the creative powers of every person and of every people, confidence in their capacities to understand, develop and use mathematics. At least three types of strategies to produce such confidence have to be considered.

A. Cultural strategies

Colonization has implied the underdevelopment of Mozambique as of most Third World countries. Underdevelopment is not only an economic process. Foreign domination also caused mathematical underdevelopment: African and American-Indian mathematics became ignored or despised; the mathematical capacities of certain peoples were negated or reduced to rote memorization; mathematics was presented as an exclusively white men’s creation and ability. Of the struggle against the mathematical underdevelopment and the combat against racial and colonial prejudice, a cultural reaffirmation makes a part. It is necessary to encourage an understanding that:

1) The people have been capable of and will be capable of developing mathematics

Examples:

a) In coastal zones of Mozambique fish is dried to be sold in the interior. How should the fish be dried? What if you place the fish at different distances from the fire? Some fish will be grilled, while others are kept wet. Through experience, the fishermen discovered that it is necessary to place the fish equidistant from the fire. They discovered a circle-concept, constructing a circle in sand using two sticks and a rope (see Figures 2 and 3).
b) How can the rectangular base of a house be constructed? In some parts of Mozambique, peasants use the following technique. One starts by laying down on the floor two long sticks of equal length (see Figure 4). Then the first two sticks are combined with two other sticks, also of equal length (Figure 5).
Now one moves the sticks until the closure of the quadrangle (one obtains a parallelogram: Figure 6). One further adjusts the figure until the diagonals are of equal length (so a rectangle is obtained: Figure 7). Now where the sticks are lying on the floor lines are drawn and the building of the house can start.

This empirical knowledge of the peasants has been propagated on a national scale, during the rabbit-raising campaign, for the construction of hutches.

2) The people’s mathematics can and will enrich the understanding of mathematics, its education and its history

Examples:

a) In the north of Mozambique, traditional villages are structured in a “circular” way (Figure 8). This village structure and e.g. the
structure of a bird’s nest led to the formation of one circle-concept “to belong to” (already a high-level abstraction). The concept is extended by comparison of form or content (“belonging to”) to e.g. the border of a basket (Figure 9).

![Circular winnowing basket](image9)

Figure 8

Figure 9
The fabrication of a sisal mat leads to the formation of another circle-concept (“spiral-circle”: Figure 10). We have still a third circle-concept, that of the fishermen. In the official language of Mozambique, Portuguese, there are two “circle” concepts, one referring to the area and the other to the circumference of a circle. By interference between the mother tongue and Portuguese, the medium of instruction, we have the following matrix (Figure 11).

![Rolling up a string of sisal](image)

**Figure 10**

<table>
<thead>
<tr>
<th>Mother tongue</th>
<th>Portuguese</th>
<th>M₁</th>
<th>M₂</th>
<th>M₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>P₁</td>
<td>P₁ = M₁</td>
<td>P₁ = M₂</td>
<td>P₁ = M₃</td>
<td></td>
</tr>
<tr>
<td>P₂</td>
<td>P₂ = M₁</td>
<td>P₂ = M₂</td>
<td>P₂ = M₃</td>
<td></td>
</tr>
</tbody>
</table>

**Matrix of language interference: circle-concepts**

![Figure 11](image)

So six different, rather complicated, situations may be encountered: now I have to use the third word (= M₃) in my
mother language but the second word \(( = P_2)\) in Portuguese ... , etc. How can this be avoided? Comparing the Portuguese “circle” concepts with the square concept, we can see that there is no need to have more than one “circle” concept: it is possible to speak about the area of a square and about the circumference of a square using the same work “square”. Paradoxically, by abolishing one of the two words in Portuguese, one enriches the scientific language: only one “circle” concept is retained and so one facilitates the learning processes of the students in this multilingual context.

b) In some regions of Mozambique, there exists a “cylinder” concept that can be described as a rolled-up rectangular mat (see Figure 12). This concept can be used as a way to discover a formula for the volume of a cylinder. It can be used not only in education, but also in formulating hypotheses on how long ago the first (approximating) formulas for the volume of cylinders could have been discovered [cf. Gerdes, 1985 b].

![Figure 12](image-url)

3) *Every people is capable of developing mathematics*

This can be shown by the cultural history of mathematics. By public lectures and by the publication of booklets on African, Indian, Chinese and Arabic mathematics, attention is drawn to the fact that many people have contributed to the development of mathematics. In this manner an attempt is made to combat a Eurocentric and distorted
vision of the history of mathematics. The booklets try to analyse, in a way accessible to a broad public, not only how, but also why and for whom, mathematics was developed in different societies at distinct times [see Gerdes 1981a, 1984a].

B. Social strategies

Within colonial society, as in every class society, prejudices about the mathematical talents of discriminated and exploited social strata and of women are widespread. For mathematics education to become emancipatory, it is necessary to encourage an understanding that:

*Children of all social classes and of both sexes have been capable and will be capable of mastering, developing and using mathematics.*

By means of counterexamples, ranging from Hypathia of Alexandria and Gauss to the winners of our National Mathematics Olympiads, sexist prejudices as well as those against children of peasants and factory workers are demystified. Special awards and scholarships are attributed to the best-qualified girls in the Mathematics Olympiads [see Gerdes 1984b].

C. Individual-collective strategies

All the aforementioned strategies are already individual in the sense that they enhance personal self-confidence by stimulating the individual’s relatedness to mathematics through the problemising reality approach, by supporting the comprehension of the relevance of mathematics, and by cultural and social self-confidence. More specific – say individual-collective strategies – are presented here by examples drawn from teaching experiences.
a) Reflection on errors

Usually “solutions” of problems are collected and presented to the students, e.g., the following two solutions of the same problem:

1) \[ \frac{\sqrt{5}+1}{\sqrt{5}-2} = \frac{\sqrt{5}+1\cdot\sqrt{5}-2}{\sqrt{5}-2\cdot\sqrt{5}-2} = \frac{\sqrt{5}+1\cdot\sqrt{5}-2}{(\sqrt{5}-2)^2} = \frac{\sqrt{5}+1\cdot\sqrt{5}-2}{9+4\sqrt{5}} = \frac{(1-2)\sqrt{5}}{9+4\sqrt{5}} = \frac{-\sqrt{5}}{9+4\sqrt{5}} \]

2) \[ \frac{\sqrt{5}+1}{\sqrt{5}-2} = \frac{(\sqrt{5}+1)(\sqrt{5}-2)}{(\sqrt{5}-2)^2} = \frac{3-\sqrt{5}}{9-4\sqrt{5}} = \frac{1-\sqrt{5}}{3-4\sqrt{5}} = \frac{1-1}{3-4} = \frac{0}{-1} = 0. \]

The students have to analyse these trial solutions, individually. Which transitions are right, which are wrong and why? In a second stage, they are asked to compare their analyses at group level and to try to achieve a common improved examination. The groups report their findings in a classroom session for a final analysis.

It is interesting to see that the students like this type of “exercise”, because it urges them to reflect, argue and rethink

b) Reflection on concept building

The students know the following definition:

When \( b \in \mathbb{R}_0^+ \) and \( n \in \mathbb{N} \), we define \( \sqrt[n]{b} \) as the positive solution in \( \mathbb{R} \) of the equation \( x^n = b \).

We ask them to debate questions such as the following:

Is it possible to define \( 0^{\sqrt{b}} \)? For which numbers? Does it make any sense?

Is it possible to define \( 1^{\sqrt{b}} \)? Is it necessary to do so?

Is it possible to define \( -3^{\sqrt{b}} \)? If so, calculate \( -3^{\sqrt{8}} \).
The questions are open-ended. E.g., in the last case, one finds answers by analogy: \(-3\sqrt{b}\) is the positive solution of the equation \(x^{-3} = b\) or by a supposed property: \(-3\sqrt{b} = \frac{1}{\sqrt[3]{b}}\).

This reflection stimulates further debate and so provides a more profound understanding of the original concept.

c) \textit{Learning to discover by discovering together}

What will be the derivative function of \(f: \mathbb{R}^+ \to \mathbb{R}\), defined by \(f(x) = \sqrt{x}\)?

The students know:
\[
\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}
\]

How do we calculate \(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\)?
How do we divide \(\Delta y\) by \(\Delta x\)?

We have \(\sqrt{x + \Delta x} = \ldots\)?
\[
\sqrt{x + \Delta x} = \sqrt{x + \sqrt{\Delta x}}, \text{ Pedro suggests.}
\]
Is it true?

Another suggestion by Lázaro: Square all the terms:
\[
(\sqrt{x + \Delta x} - \sqrt{x})^2 = \ldots
\]
Not that way? Why?

Ferdinand: Use another notation
\[
\sqrt{x + \Delta x} - \sqrt{x} = (x + \Delta x)^{\frac{1}{2}} - x^{\frac{1}{2}}
\]
How do we continue?
\[
(x + \Delta x)^{\frac{1}{2}} - x^{\frac{1}{2}} = (x + \Delta x)^{2^{-1}} - x^{2^{-1}} = \left((x + \Delta x)^{2} - x^2\right)^{-1}
\]
Why do you want to work with squares?
\[
(\sqrt{x + \Delta x})^2 - (\sqrt{x})^2 = (x + \Delta x) - x = \Delta x
\]
How do we arrive there?
In general

\[ a^2 - b^2 = \ldots \, ? \]

\[ a^2 - b^2 = (a - b) \cdot \ldots \, ? \]

\[ a^2 - b^2 = (a - b)(a + b) \]

And now every student advances:

\[
\frac{\Delta y}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \frac{(\sqrt{x+\Delta x} - \sqrt{x})(\sqrt{x+\Delta x} + \sqrt{x})}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})} =
\]

\[
= \frac{(\sqrt{x+\Delta x})^2 - (\sqrt{x})^2}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})} = \frac{(x + \Delta x) - x}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})} = \frac{\Delta x}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})} =
\]

\[
= \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}
\]

and obtains the end result

\[ f^1(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \]

By discovering together and reflecting on their process of discovery, the students enlarge their creative powers and gain self-confidence. They learn to understand the non-tautological nature of mathematical knowledge [for a more profound analysis, see Gerdes, 1985 a].

**Concluding remarks**

A problemising reality approach as starting point is in itself already a confidence-creating activity. Problemising reality, reinforced by cultural, social and individual-collective confidence-stimulating activities will contribute substantially to an emancipatory mathematics education, to enable everyone and every people to understand, develop and use mathematics as an important tool in the process of understanding reality, the reality of nature and of society, an important tool to transform reality in the service of an ever more human world.
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Chapter 13

African Slave and Calculating Prodigy:
Bicentenary of the Death of Thomas Fuller *

Abstract

Thomas Fuller (1710-1790) was an African, shipped to America as a slave in 1724. He had remarkable powers of calculation, and late in his life was discovered by antislavery campaigners who used him as a demonstration that blacks are not mentally inferior to whites. This paper describes what we know of Fuller, discusses the various uses made of his story since his death, and appeals for further study of the 18th-century African ethnomathematical context.

Thomas Fuller (1710-1790) était un Africain, emmené comme esclave en Amérique en 1724. Il avait des capacités remarquables en calcul mental, et tard dans sa vie il a été découvert par des militants anti-esclavage, qui l’utilisaient pour démontrer que l’homme noir n’est pas mentalement inférieur par rapport à l’homme blanc. Cet article décrit ce que nous savons de Fuller, discute les différents usages de son histoire, faites après sa mort, et lance un appel à plus d’études sur le contexte africain ethnomathématique du XVIIIème siècle.

In most discussions of people with extraordinary powers of mental calculation, there is brief mention of a black calculator by the name of Thomas Fuller. It is interesting to discover how he has come to join the assorted ranks of recorded calculating prodigies, for his story is located where black history, American history, and the history of African mathematics meet the history of public perceptions of mathematics and mathematicians. Just as during his life, the story of Thomas Fuller since his death 200 years ago has been one of people using him for various purposes of their own.

First, let us trace back to see how Fuller’s story has come down to us. We can start with any modern account, for example, J. Fey and J. W. Alexander’s paper bout calculating prodigies:

Thomas Fuller, brought to Virginia as a slave in 1724, could multiply two nine-digit numbers, state the number of seconds in a given period of time, and calculate the number of grains of corn in a given mass even though he never learned to read or write. [Fey and Alexander 1969, 160].

Fey and Alexander took their information from the essay by the Cambridge don W. W. Rouse Ball (1850-1925) which appeared in James R. Newman’s *The World of Mathematics* [Rouse Ball 1956, 478], reprinted from Ball’s amiable collection of *Mathematical Recreations and Essays* written earlier this century [Rouse Ball 1917, 252]. In its turn, Rouse Ball’s account is based, as is the description and analysis by Fred Barlow [1951, 17-18, 155], on papers in *The American Journal of Psychology* by F. D. Mitchell [1907, 62-63] and E. W. Scripture [1891, 2-3, 34, 58]. The latter is the source for Fuller’s appearance in the somewhat unexpected context of a tome on psychical research [Myers 1903 I, 80]. Scripture’s article and several earlier accounts of Fuller [Anonymous 1792, 265; Brissot de Warville 1794 I, 243; Stedman 1796 II, 260-261; Grégoire 1810, 183-185; Needles 1848, 32; Hoefer 1857; Williams 1883, 398-400] derive from only two independent original sources, the account written by Dr. Benjamin Rush [Rush 1789; also Dickson 1789, 185-187] and Fuller’s obituary in a Boston journal [Anonymous 1790]. We reproduce these documents in full in Appendixes 1 and 2.
On examining the contexts in which Fuller’s story has been retold and handed down over the past 2 centuries, and the emphases placed on different aspects of the sources, we can distinguish three broad phases, which may be called the liberatory, the psychologistic, and the mathematical. Again we take these in reverse time order. It is noticeable that in recent years Fuller’s black significance has been lost sight of; he does not feature, for example, in either of the recent standard works V. Newell et al. [1980] or Claudia Zaslavsky [1973]. The emphasis in the most recent phase, from Rouse Ball’s work onward – in [Smith 1983, 178-80], for example – has been simply on the comparative efficiency of Fuller and others as mathematical calculators. Thus Rouse Ball wrote of Fuller, “although more rapid than Buxton, he was a slow worker as compared with some of those whose doings are described below” [Rouse Ball 1917, 252]. Judgements of this kind are made in a context where the topic of concern is the concept and analysis of the mathematical processes of “mental prodigies.” Barlow observed in his preface, “This book has been written to satisfy the curiosity of the ever-growing public interested in abnormal mentalities and their practical manifestations” [Barlow 1951, 9]. In this sense such studies overlap with the interest shown during this century, for educational and other reasons, in the process of mathematical discovery and conceptual development, of which the works by Henri Poincaré [1908], Jacques Hadamard [1945], and Jean Piaget [1952] are well-known examples.

The seminal paper by E. W. Scripture (1864-1945) from which much of this century’s knowledge of Fuller and other arithmetical prodigies immediately derives [Scripture 1891] was, however, written in a different context. It appeared the year before Scripture was appointed to head the psychological laboratory at Yale University. He was dedicated to developing psychology as an experimental science, which he took to mean achieving a mathematization akin to that of physics, an ambitious quest as may be seen from his remark, “Experimental psychology can never rise above a rather amateurish level till the leaders can handle vectors, Hamiltonians, potentials, as well as the representatives of the physical sciences” [Thomson 1968, 143]. Although in his article of 1891 Scripture lid draw some conclusions that bear on mathematics education, the overall
perspective of the paper is of building up a collection of empirical facts preparatory to a psychological analysis of mental processes. It is noteworthy, too, that the French psychologist who showed the most interest in Fuller was the founder of experimental psychology in France, Alfred Binet [1894, 4-5], later to develop a metrization of intelligence. Binet had a noticeably more critical approach to the data, drawing attention to suspicious inconsistencies in details of Fuller’s reputed calculations. This phase of using Fuller as an empirical fact within a scientization of psychology reached its apogee in Myers’ massive *Human Personality and Its Survival of Bodily Death*. F. W. H. Myers (1843-1901) was a founder of the Society for Psychical Research, which aimed to place the study of spiritual and paranormal phenomena on an empirical scientific basis, much as Scripture and Binet intended for psychology. It was as part of his study of supernormal human faculties that he turned to the work of Scripture and Binet on arithmetical prodigies, and proffered the clearest explanation of the appeal of Fuller and the rest for this kind of investigation:

For the purpose of present illustration of the workings of genius it seems well to choose a kind of ability which is quite indisputable, and which also admits of some degree of quantitative measurement. I would choose the higher mathematical processes, were data available; […]

Meantime there is a lower class of mathematical gift, which by its very specialization and isolation seems likely to throw light on our present inquiry. [Myers 1903, 78-79]

The psychologistic approach still periodically surfaces; for instance, a recent study goes far beyond the available evidence in attempting to assess Fuller’s state of mind by 20th-century criteria: “No IQ tests or other clinical descriptions are available to precisely establish his intellectual level, but the information available certainly points toward serious mental handicap” [Treffert 1989, 56].

The context in which Fuller’s abilities first came to light, however, was completely different from the concerns of scientific psychology which arose a full 100 years after his death. The early writers and commentators were all working in, or sympathetic to, the
antislavery movement. We only know of Fuller, indeed, because people wanted to use him as evidence in the abolitionist cause. For the earliest account of Fuller (Appendix 1) we are indebted to Benjamin Rush (1745/1746-1813). Rush was a man of remarkable energy – physician, psychiatrist, professor of chemistry, signatory to the Declaration of Independence, and ceaseless worker for a host of radical causes – and was Secretary to the Pennsylvania Society for the Abolition of Slavery when word reached Philadelphia of the remarkable abilities of this slave living in Virginia. Rush wrote up an account 1 of the visit paid to Fuller by two members of the society, William Hartshorne and Samuel Coates, to send to the British hub of the movement. The reason for this is clear from Rush’s covering letter to an early report of Fuller’s abilities [Rush 1788, 306], as well as from the note, which accompanied Rush’s account in the *American Museum*:

The abolition society in London, having requested the society for the abolition of slavery in Philadelphia, to transmit to them such accounts of mental improvement, in any of the blacks, as might fall under their notice, in order the better to enable them to contradict those who assert that the intellectual faculties of the negroes are not capable of improvement equal to the rest of mankind, these certificates were accordingly forwarded to London, with the society’s last letters, in addition to others heretofore sent. [Rush 1789, 62]

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1 Rush’s letter to the London Society about Fuller (Appendix 1) is not reproduced in the two-volume Princeton edition of Rush’s letters [Rush 1951], the editor of which was presumably faithful to his own conception of significance in interpreting his principle of selection:

The intention has been to print all the available letters that have some literary distinction or that tell something significant about the writer or his times. [Rush 1951, Introduction]
Fuller was, then, a clear demonstration that black people were not to be distinguished from others by intellectual criteria; under the circumstances of the slave trade, he was one of the few documented demonstrations of this.

Why was such a demonstration necessary? A common justification for slavery was that blacks were intrinsically, naturally, inferior to whites [Barker 1978]. This view was promoted in the works of philosophers who influenced the thinking of the age. John Locke (1632-1704), for example, had showed how by reflecting on differences of colour a child might “demonstrate by the Principle, It is impossible for the same Thing to be, and not to be, that a Negro is not a Man” [Locke 1690, 607] – a significant choice of example [Caffentzis 1989, 193-202]. And notwithstanding his radical free-thinking reputation, David Hume (1711-1776) inserted is footnote to one of his essays:

I am apt to suspect the Negroes to be naturally inferior to the Whites. There scarcely ever is a civilized nation of that complexion, nor even any individual, eminent in either action or speculation. No ingenious manufactures amongst them, no arts, no sciences. … Such a uniform and constant difference could not happen, in so many countries and ages, if nature had not made an original distinction between these breeds of men. Not to mention our colonies, there are Negro slaves dispersed all over Europe, of whom none ever discovered any symptoms of ingenuity, though low people, without education, will start up amongst us, and distinguish themselves in every profession. In Jamaica, indeed, they talk of one Negro as a man of parts and learning; but it is likely he is admired for slender accomplishments, like a parrot who speaks a few words plainly. [Hume 1741-1742, 123]

So blunt an assertion provided a guide for abolitionists in suggesting how to set about refuting the argument, and thus knock out a linchpin of the proslavery ideology: find a counterexample – a black of parts, learning, or ingenuity. But as Hume’s comments on the Jamaican make clear, a reported counterexample could simply be
denied or laughed away if it clashed with one’s preconceptions. It is in this context that we can better understand Rush’s concern to point out that Hartshorne and Coates were “men of probity and respectable characters” [Rush 1789], and his formally certifying the truth and faithfulness of his report [Dickson 1789, 187]. Establishing the reliability of the testimony was crucial to the purpose it was intended to serve; the abolitionists were not interested in the information about Fuller for academic reasons, as it were, and it would be useless if discounted as unreliable or exaggerated. But if accepted as properly attested, it would be a powerful weapon in serving the cause.

And so it proved; it is just this aspect that is picked up by the writers who cited Fuller over the next few years and beyond. News of his abilities spread fast through the antislavery community, and we find a shorter account by Rush in an Oxford tract [Burgess 1789, 132-133, 149-150], and his longer account [Appendix 1] reproduced verbatim, the same year, in William Dickson’s passionate Letters on Slavery [Dickson 1789, 185-7]. 2 The story had already appeared in the monthly magazines, in the Universal Magazine for December 1788, and the Gentleman’s Magazine for the same month. The latter pointed out the moral in terms that its readers could understand:

This is the more extraordinary as it is somewhere remarked that few of the race of woolley-headed blacks

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2 The antislavery tract by Dickson, who had been private secretary to the Governor of Barbados, is interesting to us in making brief mention of a Negro Poet and Mathematician by the name of Francis Williams; little about Williams’ mathematics has come to light yet, but the sample of his verse on Dickson’s title page [Dickson 1789] should have sufficed to remove all doubts of black mental capacity:

Pollenti stabilita manu (Deus almus eandem Oigenis animam, nil prohibente dedit)

Ipsa coloris egens virtus, prudentia; honesto Nullus inest animo, nullus in arte color.
can go farther in the art of enumeration than the number 5. [Anonymous 1788, 1112]³

At this period there was much French interest too. J. P. Brissot de Warville (1754-1793), for example, the French revolutionary politician who toured North America in 1788 as representative of the Société des Amis des Noirs, noted:

It has been generally thought, and even written by some writers of note, that the Blacks are inferior to the Whites in mental capacity. This opinion begins to disappear; the Northern States furnish examples to the contrary. I shall cite two, which are striking ones: the first [James Derham, a black physician] proves that, by instruction, a Black may be rendered capable of any of the professions: the second [Thomas Fuller], that the head of a Negro may be organized for the most astonishing calculations, and consequently for all the sciences. [...] These instances prove, without doubt, that the capacity of the Negroes may be extended to anything; that they have only need of instruction and liberty. [1794 I, 243]

Another French revolutionary, who cited Fuller in this context, in his efforts to ensure equal rights for black and white in French colonies, was the Abbé Henri Gregoire (1750-1831) [1810, 183-185]. And the most eloquent location of Fuller within this argument came later in the 19th century, when the first black member of the Ohio Legislature wrote his History of the Negro Race in America:

One of the standing arguments against the Negro was, that he lacked the faculty of solving mathematical problems. This charge was made without a disposition to allow him

³ The writer had evidently not read Barbot [1732, 414], which gives the counting words from one to a thousand, in various languages found along the Slave Coast (Togo and Benin), showing various sophisticated numerations on a basically 5-20 system.
an opportunity to submit himself to a proper test. It was equivalent to putting out a man’s eyes, and then asserting boldly that he cannot see; of manacling his ankles, and charging him with the inability to run. But notwithstanding all the prohibitions against instructing the Negro, and his far remove from intellectual stimulants, the subject to whom attention is now called [Thomas Fuller] had within his own untutored intellect the elements of a great mathematician. [...] That he was a prodigy, no one will question. He was the wonder of the age. [Williams 1883, 398-399]

This vigorous rhetoric raises the question, which would surface increasingly in the psychologistic phase that followed Scripture’s work on prodigies a few years later, of how good a mathematician Fuller was: did he indeed possess “the elements of a great mathematician”? One obvious fact to emerge from the later comparative work on calculating prodigies is that some – the mathematical Mozarts, as it were – do turn into, or are already, notable mathematicians: Wallis, Gauss, Ampere. At the other extreme, there are arithmetical prodigies of the idiot savant kind, who combine this single remarkable skill with no other apparent mathematical insight or creativity. It is well known, too, that popular opinion regards any mathematical activity as remarkable, and uses genius and similar words somewhat more freely than historians might feel justified [Anonymous 1913]. We cannot now tell into which category Fuller should be placed; whether he was really a mute inglorious Newton is essentially unknowable, and indeed a somewhat foolish line of inquiry. The following considerations are more helpful in reaching an understanding of the significance of Thomas Fuller.

Fuller was about 70 years old when he was visited by Hartshorne and Coates, and correctly answered all their questions. His memory was already failing, by his own account [Rush 1789, 62], and we have no information about what he was capable of when in his prime, still less what he would have been capable of under other circumstances. In particular, there is insufficient evidence for the judgments made by analysts in the later phases of inquiry, who felt the need to classify Fuller and the other prodigies in various ways: Rouse Ball spoke of
him as a relatively “slow worker” [1917, 252], and Myers classified his intelligence as “low” [1903, 80] – a claim which is belied by his final response to Coates [Rush 1789, 62]. Fuller was brought to America when he was 14 years old [Anonymous 1790, 123], that is, in about 1724. Most probably he had already developed his mental calculation abilities by then; certainly his exceptional abilities cannot be understood except through closer examination of the cultural context that stimulated their development.

This poses the question of Fuller’s African education: his learning of number words, of a numeration system, of arithmetical operations, of riddles and mathematical games, and so on. The extant evidence on this in the early 18th century is as yet not great, although we do know of an astronomer and mathematician of that period, Muhammad ibn Muhammad, from the Fulani people of Katsina (now northern Nigeria) [Zaslavsky 1973, 138-151], of whom an Egyptian contemporary wrote, “He acquired this learning in his own country” [Zaslavsky 1973, 151]. Among the sparse other evidence for 18th-century sub-Saharan mathematical practice which we have at present, the abilities of the inhabitants of Fida (on the coast of Benin) were attested by John Barbot:

The Fidasians are so expert in keeping their accompts, that they easily reckon as exact, and is quick by memory, as we can do with pen and ink, though the sum amount to never so many thousands: which very much facilitates the trade the Europeans have with them; and is not half so troublesome, as with other Guineans, who are commonly very dull on this head. [Barbot 1732, 339]

Later in the century, the fervent antislavery crusader Thomas Clarkson (1760-1846) gave a fuller comparative account of the calculating abilities of African slave-traders:

It is astonishing with what facility the African brokers reckon up the exchange of European goods for slaves. One of these brokers has perhaps ten slaves to sell, and for each of these he demands ten different articles. He reduces them immediately by the head into bars, coppers, ounces,
according to the medium of exchange that prevails in the part of the country in which he resides, and immediately strikes the balance.

The European, on the other hand, takes his pen, and with great deliberation, and with all the advantages of arithmetick and letters, begins to estimate also. He is so unfortunate often, as to make a mistake; but he no sooner errs, than he is detected by this man of inferiour capacity, whom he can neither deceive in the name or quality of his goods, nor in the balance of his account. Instances of this kind are very frequent: and it is now the general complaint of the captains sent upon the coast, that the African brokers are so nice in their calculations, that they can scarcely come off with a decent bargain.

I presume that instances of this kind will be received as proofs of the existence of their understandings, all arithmetical calculations being operations of the mind. [Clarkson 1788, 125]

It is of course possible, as Scripture [1891, 2] suggests, that these mental computational abilities of African slave-dealers were “brought to the front or produced by the necessity of competing with English traders armed with pencil and paper”; but this judgement shows an over-willingness to ignore the social context of the traders’ lives. Their abilities could have been brought to the fore only if there existed a fertile cultural ground for their development, just as in the case of Fuller’s talents.

Although it may be that the slave trade stimulated in some way the powers of mental calculation among African brokers, it clearly influenced in a very destructive way the subsequent development of mathematics in sub-Saharan Africa. Recent research by one of us [Gerdes 1989] has shown the destructive influence of the slave trade, and colonialism in general, on the mathematical heritage embedded in the sand-drawing tradition of the Tchokwe people of North-east Angola. A systematic analysis of their standardized drawings, collected by missionaries and ethnographers in the period 1930-1955, makes explicit the mathematical knowledge of the akwa kuta sona
(professional drawing experts), especially of the properties of single closed line patterns, and has also made possible the reconstruction of their further mathematical knowledge, lost over time. Like the professional knowledge of Tchokwe smiths, the knowledge of the *akwa kuta sona* was mostly secret, transmitted from father to son. When a blacksmith or a drawing expert was taken prisoner and sold as a slave, his specific professional knowledge might disappear completely from his village or region. Thus the slave trade was extremely destructive of the development of the existing mathematical traditions and potential, through breaking the professional continuity and depriving Africa of bearers of mathematical knowledge and skill such as Thomas Fuller.

Recent ethnomathematical research in Nigeria, as summarized by Shirley [1988], shows the survival, nonetheless, of a rich tradition of mental calculations among illiterate people. It would be interesting and valuable to search for new records with data on the geographical or ethnic origin of Thomas Fuller, and to correlate this information with ethnomathematical and historical research on the same region or ethnic group. An ethnomathematical phase of investigations into Fuller would be a suitable start to the third century after his death.

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4 The gross statistics for the decade 1721-1730 show that nearly a quarter of slaves transported in that period were from the Gold Coast, and a further 20% from each of the Windward Coast (Ivory Coast/Liberia) and the Bight of Benin (Volta to Benin) [Curtin 1971, 367]. So Fuller was probably one of the 150,200 slaves shipped in the 1720s from somewhere between Liberia and Benin. In that decade only 2% of slaves came from the Bight of Biafra (including the Niger Delta), although the proportion from here rose considerably later in the century.
Philadelphia, November 14, 1788

Account of a wonderful talent for arithmetical calculation,
in an African slave, living in Virginia.

There is now living, about four miles from Alexandria, in the state of Virginia, a Negro slave of seventy years old, of the name of Thomas Fuller, the property of Mrs. Elizabeth Coxe. This man possesses a talent for arithmetical calculation; the history of which, I conceive, merits a place in the records of the human mind. He is a native of Africa, and can neither read nor write. Two gentlemen, natives of Pennsylvania, viz. William Hartshorne and Samuel Coates, men of probity and respectable characters, having heard, in traveling through the neighbourhood, in which this slave lived, of his extraordinary powers in arithmetic, sent for him, and had their curiosity sufficiently gratified by the answers which he gave to the following questions.

First. Upon being asked, how many seconds there are in a year and a half, he answered in about two minutes, 47,304,000.

Second. On being asked how many seconds a man has lived, who is seventy years, seventeen days and twelve hours old, he answered, in a minute and a half, 2,210,500,800.

One of the gentlemen, who employed himself with his pen in making these calculations, told him he was wrong, and that the sum was not so great as he had said upon which the old man hastily replied, “top, massa, you forget de leap year.” On adding the seconds of the leap years to the others, the amount of the whole in both their sums agreed exactly.

Third. The following question was then proposed to him: suppose a farmer has six sows, and each sow has six female pigs, the first year, and they all increase in the same proportion, to the end of
eight years, how many sows will the farmer then have? In ten minutes, he answered, 34,588,806. The difference of time between his answering this, and the two former questions, was occasioned by a trifling mistake he made from a misapprehension of the question.

In the presence of Thomas Wistar and Benjamin W. Morris, two respectable citizens of Philadelphia, he gave the amount of nine figures, multiplied by nine.

He informed the first-mentioned gentlemen that he began his application to figures by counting ten, and that when he was able to count an hundred, he thought himself (to use his own words) a very clever fellow.”

His first attempt after this was to count the number of hairs in a cow’s tail, which he found to be 2872.

He next amused himself with counting, grain by grain, a bushel of wheat and a bushel of flax-feed.

From this he was led to calculate with the most perfect accuracy, how many shingles a house of certain dimensions would require to cover it, and how many posts and rails were necessary to enclose, and how many grains of corn were necessary to sow a certain quantity of ground. From this application of his talents, his mistress has often derived considerable benefit.

At the time he gave this account of himself, he said his memory began to fail him – he was gray-haired, and exhibited several other marks of the weakness of old age – he had worked hard upon a farm during the whole of his life, but had never been intemperate in the use of spirituous liquors. He spoke with great respect of his mistress, and mentioned in a particular manner his obligations to her for refusing to sell him, which she had been tempted to do by offers of large sums of money, from several curious persons.

One of the gentlemen (mr. Coates) having remarked in his presence, that it was a pity he had not had an education equal to his genius; he said, “no massa – it is best I got no learning; for many learned men be great fools.”
Died – NEGRO TOM, the famous African Calculator, aged 80 years. He was the property of Mrs Elizabeth Cox of Alexandria. Tom was a very black man. He was brought to this country at the age of 14, and was sold as a slave with many of his unfortunate Countrymen.

This man was a prodigy. Though he could neither read nor write, he had perfectly acquired the art of enumeration. The power of recollection and the strength of memory were so complete in him, that he could multiply seven into itself, that product by 7, and the product, so produced, by seven, for seven times. He could give the number of months, days, weeks, hours, minutes, and seconds in any period of time that any person chose to mention, allowing in his calculation for all the leap years that happened in the time; and would give the number of poles, yards, feet, inches, and barley-corns in any given distance, say the diameter of the earth’s orbit; and in every calculation he would produce the true answer in less time than ninety-nine men in a hundred would produce with their pens. And, what was, perhaps, more extraordinary, though interrupted in the progress of his calculation, and engaged in discourse upon any other subject, his operations were not thereby in the least deranged, so as to make it necessary for him to begin again, but he would go on from where he had left off, and could give any, or all, of the stages through which the calculation had passed. His first essay in numbers was counting the hairs in the tails of the cows and horses, which he was set to keep. With little instruction he would have been able to cast up plats of land. He took great notice of the lines of land, which he had seen surveyed. He drew just conclusions from facts; surprisingly so, for his opportunities. Thus died Negro Tom, this self-taught Arithmetician, this untutored scholar! Had his opportunity of improvement been equal to those of thousands of his fellow-men, neither the Royal Society of London, the Academy of Sciences at Paris, nor even a NEWTON himself, need have been ashamed to acknowledge him a Brother in Science.
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About the author

Paulus Gerdes has been a professor of mathematics at universities in Mozambique. He served as Head of the Department of Mathematics and Physics (1981-1983), as Dean of the Faculty of Education (1983-1987), and as Dean of the Faculty of Mathematical Sciences (1987-1989) of the Eduardo Mondlane University.

From 1989 to 1996 he was the President (Vice-Chancellor) of the Pedagogical University, with branches in the South, Centre and North of Mozambique. He was a visiting professor at several universities in Brazil, France, Peru, South Africa, and the United States.

Since 1998 Professor Paulus Gerdes has been Director of the Mozambican Ethnomathematics Research Centre in Maputo. From 2000 to 2004 he was the senior advisor to the Minister of Education. In 2006-2007, he was president of the founding commission of the Lurio University, Mozambique’s third public university established in Nampula in the north of the country. Currently, he is senior advisor for research and quality of education at the ISTEG-university in Boane, in the south of Mozambique.

Among his international functions it may be mentioned that Dr. Paulus Gerdes was, from 1986 to 2013, Chair of the African Mathematical Union Commission for the History of Mathematics in Africa (AMUCHMA). He was President of the International Association for Science and Cultural Diversity (2001-2005). From 1991 to 1995 he was Secretary of the Southern African Mathematical Sciences Association (SAMSA). He was a member of the Executive Committee of the African Mathematical Union (1986-1995, 2000-2004). Since 2000, he has been President of the International Study Group for Ethnomathematics. He is a fellow of the African Academy of Sciences (AAS) and of the International Academy for the History of
Science. He was elected, in 2005, and re-elected in 2011, AAS Vice-President, responsible for the Southern African region.

In 2012 Professor Gerdes received the highest distinction “Prize for Excellence in Higher Education (Research and Teaching)” on the occasion of the commemoration of 50 years of Higher Education in Mozambique.
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