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ON THE EPISTEMOLOGICAL LIMITS OF LANGUAGE: MATHEMATICAL KNOWLEDGE AND SOCIAL PRACTICE DURING THE RENAISSANCE*

ABSTRACT. An important characteristic of contemporary reflections in mathematics education is the attention given to language and discourse. No longer viewed as only a more or less useful tool to express thought, language today appears to be invested with unprecedented cognitive and epistemological possibilities. One would say that the wall between language and thought has crumbled to the point that now we no longer know where one ends and the other begins. At any rate, the thesis that there is independence between the elaboration of a thought and its codification is no longer acceptable. The attention given to language cannot ignore, nevertheless, the question of its epistemological limits. More precisely, can we ascribe to language and to the discursive activity the force of creating the theoretical objects of the world of individuals? In this article, I suggest that all efforts to understand the conceptual reality and the production of knowledge cannot restrict themselves to language and the discursive activity, but that they equally need to include the social practices that underlie them. This point is illustrated through the analysis of the relationship between mathematical knowledge and the social practice of the Renaissance.

KEY WORDS: Cognition, discourse, epistemology, language, mathematical thinking, meaning, Renaissance mathematics, semiotics, signs, social practice

1. THE PARADIGM OF LANGUAGE

Without a doubt, contemporary social science research is currently living through what Markus called "the paradigm of language" (Markus, 1982). With this term he designated the paradigm which conceives of man's (sic) relationship to his world as having an analogous relationship to modes of linguistic functioning. Thus, for example, man's relationship to his world can be seen as a universal form of social relationships functioning according to the modalities of a kind of ongoing communication (Gadamer). The relationship between man and his world can also be seen as an homogenous structural system existing behind acts of speech and transcending the locality and historicity of human actions (as in Lévi-Strauss' and Saussure's structuralism). At any rate, regardless of the form

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it takes, this paradigm sees man's relationship to the world as passing through forms of social interaction modeled on a distinctive conception of language. Its central question is neither to attempt to understand the place of man in God's creation nor is it to investigate the limits of human reason in the attainment of objective knowledge (Kant). The question is now to grasp the possibilities of discourse and to understand man as *homo dialogicus*. It seems that the age of Judgment inaugurated by Kant arrives to its end and we now enter into the age of Communication.

One important idea, brought to the fore by the debates surrounding this paradigm is the need to reconsider what is meant by *cognition*. The paradigm of language claims that it is inadequate to understand cognition as an ensemble of strictly mental processes, as proposed by classical cognitivism, and urges us to see instead cognition as an activity that goes beyond the confines of the individual's skull (Bateson, 1973; Geertz, 1973; Edwards, 1997; Myerson, 1994). In this new paradigm, language acquires a new dimension – an epistemological one[1].

The new paradigm endows language (and more generally, discourse) with an unprecedented epistemological importance, to the extent that, in many cases, it is considered (implicitly or explicitly) as the arena where knowledge and its meaning are constituted.[2]

In the past few years, the influence of the paradigm of language has become apparent in mathematics education. Irrespective of the variety of possible conceptualizations of the relationship between thought and language[3], at the heart of many current didactic reflections, we find a marked tendency to consider language and discourse as the producers of knowledge and ideas. Nevertheless, we are justified in asking the following question: Can we truly ascribe to language this power of creating the theoretical objects of the individual's world? There are reasons to be skeptical. As Leont'ev cautions us, language is not a demiurge: "words, the language signs, are not simply replacements for things, their conditional substitutes" because behind their meanings, adds Leont'ev "is hidden social practice [that is] activity transformed and crystallized in them". (Leont'ev, 1978, p. 18).

Certainly, a word of discourse which, according to the paradigm of language, would allegedly give birth to a conceptual object is not a word invented on the spot. The meanings of words and symbols retain, in crystallized forms, the activities of individuals. To cite just one example, behind the history of the word 'counter' one finds the merchants' actions of *counting* dating from the High Middle Ages. The word 'counter' has its roots in this practice of counting, taking place on a *table*, where the merchants placed tokens and other objects, used as guides to help their count[4]. In actual fact, words have a cultural history behind them. Bakhtin said that,

> . . . the topic of the speaker's speech, regardless of what this topic may be, does not become the object of speech for the first time in any given utterance; a given speaker is not the first to speak about it. The object, as it were, has already been articulated, disputed, elucidated, and evaluated in various ways. Various viewpoints, world views, and trends cross, converge, and diverge in it. The speaker is not the biblical Adam, dealing only with virgin and still unnamed objects, giving them names for the first time. Simplistic ideas about communication as a logical-psychological basis for the sentence recall this mythical Adam.
> (Bakhtin, 1986, p. 93)

The word which we utter and the object to which it refers are inevitably tied to an historically constituted social practice. Indeed, this social practice, which is the niche of the objectifying word of the conceptual reality of individuals, establishes and delineates the way the word functions, even *before* the word has actually been uttered.

By making such observations, I am not trying to challenge the role of language in the constitution of knowledge, but rather to call attention to the problem of its own epistemological limits. It seems to me that to reify words and discourse would be to forget that the object of knowledge, the subject that seeks knowledge and the relationship between the subject and the knowledge it seeks are all subsumed in a much wider network of historical and cultural significations than first assumed.

We may need to expand the current understanding of language and discourse and to subsume it into a more general concept of semiotic system, which considers artefacts, words and symbols as key elements in the organization of mental processes as they are used to reflect and objectify ideas in the course of the individuals' activities (Radford, 1998, 2003a). In this line of thought, cognition appears as definitely consubstantial with the various forms of social organization, types of production and cultural modes knowing. As Wartofsky noted,

> . . . the nature of knowing, of cognitive acquisition itself, changes historically; *how* we know changes with changes in our modes of social and technological practice, with changes in our forms of social organization. (Wartofsky, 1979, p. xiii)

Although there are several trends in psychology which lay emphasis upon the relationship between social practice and knowledge[5], we can say that despite some notable exceptions[6], there are still few works which explore the problem of language, mathematics, and cognition from an epistemological point of view, and even fewer which try to define the limits of language[7]. The avenue that I have chosen in order to explore these limits (limits conceived of not only in their negative aspects, but in their positive aspects as well), is to situate language at the heart of social practice and the cultural construction of knowledge. While it may be true, as argued by Eagleton (1983, p. 175), that all thought is restricted by language, one still needs to explain how topics become socially relevant and how the individual's discourse thematizes them. Knowledge, as Foucault said, is not disinterested. For instance, it is well known that Plato and his disciples did not talk about the arithmetic enigmas that the Babylonians and Egyptians solved using false-position methods, or methods that today we might call algebraic. It would be difficult to argue that Plato was not aware of the mathematical practices of the Orient, and it would be misleading to state that Plato and his disciples did not deal with Babylonian and Egyptian enigmas because of a lack of appropriate language[8]. It was a matter of selection, and this selection was not made capriciously. Rather, it resulted, to an important extent, from the Greek logic of cultural significations (Radford, 2003b).

The goal of this article is to offer an epistemological exploration on the relationship between language, knowledge and cultural practice. This exploration will be centered on the mathematics of the Renaissance. I will attempt to outline the cultural configurations that made these mathematics possible, while at the same time allowing for their limits. Sections 3, 4 and 5 make up the heart of the article and it is in these sections that I discuss the abacists' mathematical thinking. Section 3 offers some elements of the abacist epistemology, elements which will be taken up again in section 4, in terms of the semiotics of mathematical representation. My effort is directed at understanding the 'mode of reason' of abacist research, in order to then correlate it to the word of discourse and the sign of the objectifying action, and the conceptual object thus objectified. In order to better grasp the accomplishment of abacist thinking, section 5 is devoted to briefly comparing the use of proportion by abacists and the Greek geometers. As my objective is to scrutinize mathematical thought and discourse in its historical-cultural context, section 2 is devoted to the cultural configurations in which the abacist school and masters found themselves historically immersed.

2. The Abacist school and its masters in their historical context

2.1. *The historical-cultural configuration*

The historian of mathematics Gino Arrighi, after a long career spent editing and analyzing mathematical texts from the High Middle Ages to the Renaissance, described what he felt would most surprise a first time reader of an abacist text. He says:

> Examining any abacist book one remains surprised when realizing how spread out and with what variety the 'exchange of merchandise' is treated. It's the same

thing on the quasi totality of the abacist [treatises]: 'two [persons] barter wool and cloth' is the way Pier Maria Calandri begins his every 'ragione' or exercise of exchange. (Arrighi, 1992, p. 10)

To account for the frequency of these problems on exchange, he goes on to explain:

The presence of barter/exchange is an indication of a natural or guided insufficiency of currency and the abacists compiled those spread out items in order to facilitate the merchants in their calculations during difficult times.
(Arrighi, 1992, p. 10)

The main forms of property during the feudal period were land and the serfs working on it, and sometimes the work of the individual (e.g. the craftsman) and the small amount of capital it generated. These forms of property remained organized according to the restrictions imposed by the precarious state of agricultural conditions or the simplicity of the varieties of craft industries. However, during the emergence of capitalism, the fundamental form of property became work and trade. In *The German Ideology* Marx and Engels (1846/1982) note that during the feudal period, 'trade' took place between the individual and the natural world, and that the profits from such trade were the products that the individual reaped from nature. On the other hand, during the emergence of capitalism, the dominant form of trade was the various forms of exchange *between individuals* themselves. Clearly, this is what is being reflected in a great number of abacist mathematical problems, among them the following problem taken from the *Trattato d'abaco* of Piero della Francesca (ca. 1416–ca. 1492), edited by Arrighi (1970):

Two individuals are bartering, one with wax and the other with wool. The wax is worth 9 ducati 1/4 and the barter rate is 10 2/3; the other one has wool and I do not know what the thousand is worth, its exchange rate is 34 ducati, and the barter was equal [i.e. fair]. How much was the wool worth cash?
(Arrighi, (ed.), 1970, p. 51)

Trade between individuals as the fundamental form of property became possible after the separation of production and commerce. This separation made commercial enterprise an activity between individuals that could go beyond the peripheries of neighbourhoods or villages, and instead reach distant cities. This then dissolved the obstacle of distance in the feudal world, and time became the uppermost factor in the organization of people's lives. Before, people had spent the day working the land for their subsistence, but now, time came to be distributed between agriculture and manufacture. Furthermore, with the stricter division of labourers (into producers, merchants, entrepreneurs, etc.) time became exclusively devoted to one of the new kinds of occupations that arose in the post-feudal period.

Let us pause to consider the arithmetic problems dealing with trade that Arrighi has brought to our attention. These problems relate commercial situations involving exchanges of merchandise, and introduce two people in possession of different goods that they would like to trade. While trade had been practiced with intensity since the first civilisations, it had never been the object of such systematic mathematical study as during the Renaissance. Such study required meticulous calculations of weights and measures, whose purpose was either to ensure an exchange in one's own favour, or to establish the conditions for fair exchange and prevent fraud. For Luca Pacioli for instance, trade meant exchanging one's own goods with the intention of getting a greater amount of goods from the other party (see Hadden, 1994, p. 90). In contrast, della Francesca felt that mathematical calculations should establish conditions under which neither party could be deceived during the exchange (this is 'fair trade' as in the problem cited above).

Nevertheless, despite its apparent simplicity, trade engenders a whole series of important abstractions. Indeed, before we can conceive of trade as an activity demanding precise numeric procedures to know how much to give of one product, in exchange for a certain amount of another, we must first know the 'usage value' of merchandise, that is to say, the value assigned to the *utility* it has in the individual's life[9].

Consequently, trade requires that we establish the relation of two usage values. But, trade goes beyond usage value itself. Indeed, what can be deduced from the fact that **x** quantity of wax is equal to **y** quantity of wool? What does this equality really establish? It establishes the *exchangeable value* of these goods, for, as Ricardo said, "utility (...) is not the measure of exchangeable value, although it is absolutely essential to it" (1817/2001, p. 8). We should interpret the preceding equality on many levels.

Firstly, we have the *qualitative* aspect: this is the qualitative relation between two objects of differing natures, the wax on the one hand and the wool on the other. After all, as Marx has already noted in Book 1 of *The Capital*, one does not exchange one suit for another identical suit. Certainly, equality depends on qualitative difference.

Secondly, we have the *quantitative* aspect: we need to know what quantity of wax is equivalent to two cloths made of wool. Equality then compares the exchangeable value of two things. However, it is neither the wax, nor the wool which makes this equality possible; rather it is something quite different. It is a new mathematical object: *value*. This is what allows us to exchange wax, not just for wool, but for other products as well.

Therefore, *the practice of trade* leads to a concatenation of important abstractions: (1) usage value, (2) exchangeable value and (3) value itself.

These abstractions are not subject to an open or explicit thematization on the part of the abacists, but rather are situated within practical knowledge as structured by the cultural institutions of Renaissance. In the following section we shall see what kind of formal structure was developed by the Italian cities, precisely to ensure the circulation of this knowledge.

2.2. *The abacus school as a social institution for the production and circulation of knowledge*

How should one exchange goods? How to keep precise accounts? In a study on the documents of Renaissance merchants and businessmen, Bec states:

> . . . one worry for merchants is how to be able to keep fair and precise accounts. Thanks to numbers, the *mercatores* can measure the universe and bring it back to human scale. In their account books, they carefully specify the weight, length, volume, surface and price of the merchandise and goods that they buy or sell.
> (Bec, 1967, p. 316)

Meticulous calculations of weights and measures that underlie the value of merchandise, the way to keep track of transactions in account books, etc., all require the establishment and circulation of a new kind of knowledge. Indeed, this was precisely the basis for creating abacus schools, which began to be set up in the 13^{th} century, the first in Bologna in 1265 (Grendler, 1989, p. 5). Italian city councils had quickly recognized the importance of a suitable education in the domain of commerce and began to hire teachers (called *maestri d'abbaco*) to instruct children. One document mentions the creation of an abacus school, in 1284, in Verona, where Maestro Lotto of Florence had been called to teach (Franci and Toti Rigatelli, 1989, p. 68). In addition to these local schools, where the curriculum and the choice of instructors fell to the city councillors, there were also independent schools run by masters who were not at all affiliated with the community council. This kind of school would often be an institution for training in commerce called a *bottega d'abbaco*, just as there were also independent schools for studying Latin (conducted by *maestri di grammatica*).

The importance of these independent schools can be clearly seen in a census completed in Venice, showing that in 1587–8, independent schools (either of Latin, the vernacular or abacus schools) serviced 89% of the city's students (Grendler, 1989, p. 43). In the case of the independent schools, it was the master himself who paid for the school's establishment, and then was remunerated according to the fees students paid for instruction. Another possible variant was that the master acted as a kind of tutor for one or more children of a particular individual for a certain number of years. By the end of their study period, the students had to be able to draft commercial legal documents, keep account books and do any other calculations required by the family business. Lastly, there were cases where a group of parents would agree to pay the same master for teaching their children. It is precisely as a tutor for the children of the merchant Antonio de Rompiasi, that Luca Pacioli began his career.

Depending on the type of school (to which schools with church affiliations can be added) there was an obvious difference in the type of instruction provided to students. Town councillors sought out the best means of attracting the most prestigious masters (sometimes finding them in neighbouring cities). This in turn resulted in a stratification in the quality of mathematical practices learned by merchants and their sons. In the introduction to his work *Summa de artithmetica geometria proportioni et proportionalita*, printed in 1494 and reprinted posthumously in 1523 Pacioli notes:

> Many merchants have old ways of figuring which they think are the best and the shortest, and which they learned from some vulgar and ignorant numberer at some crossroads instead of learning in the public squares from those chosen by the community, which tries to get the best trained men.
>
> (Pacioli, 1523; cited in Taylor 1980, p. 63)

2.3. *The abacus master*

Naturally, when Pacioli made these statements, he was no longer an abacus tutor. By this time he was an accomplished mathematician, who, in 1475, had been called to Perugia as a public lecturer in mathematics and who, in 1494 (or 1493: see Bernini, 1982) had been designated professor of mathematics at the Urbino court[10]. The abacus masters, like the Latin masters, led a life characteristic of the urban middle-class during the Renaissance (van Egmond, 1976). As a rule, they enjoyed a prestige similar to that of craftsmen (Franci, 1988, p. 183). They lived in modest homes, and sometimes owned a house in the country.

Unlike most of the craftsmen or artisans (for example painters and sculptors), who belonged to guilds which were authorized to accredit and monitor the practice of their respective professions, abacists had no professional association to supervise their members. While other craftsmen submitted to testing to prove their mastery of their profession, abacus masters had to build their reputations by practice alone. Depending on the number of students, an abacus master could call on other abacus masters to give lessons as well[11]. In this case, just as with the town schools, the choice of instructors was made carefully, in order to maintain the quality of a school, as well as its prestige. In this context, rivalries between schools were not unheard of. Franci (1988, pp. 188–89) recounts how, following the death of its director, a certain school in Florence (la Bottega d'abaco di Santa Trinità) was the subject of a plot by another school in the city that wanted to force it to close. In 1346, an abacus master was put in prison for three months, after an incident involving another abacus master, following a dispute over their right to a group of students.

It is inside this historical context that the abacist as social subject and his thought was constituted. It is useful to remember that this was a highly competitive time and one in which the abacus masters Scipione dal Ferro, Cardan and Tartaglia sought a formula to solve the cubic equation, and subsequently achieved what Franci (1988, p. 188) calls the first great innovation in mathematics of the modern period. In the next section, the cultural context previously sketched will help us understand the abacist's 'mode of reason'.

3. The 'Mode of Reason' of the Abacists

Abacist research is spread out, as Rojano (1996) has remarked, according to the axis of the *formulation* of problems. This axis is, so to speak, 'intersected' by another axis, which pertains to the method or methods used to solve the formulated problem. Although for many problems, it is simply a case of applying an established method to the given situation, in other instances, mathematical methods need to be expanded using increasingly complex technicalities. Thus, problems dealing with trade, such as the one we mentioned in section 2.1 (i.e. the problem about bartering wool and whose solution we will analyze later), are followed by other problems where one of the participants is trading partly with cash and partly with merchandise. In problems dealing with companies, after having outlined the distribution of profits according to the amount contributed by each investor, the problems become more complicated by supposing that, in addition to a certain amount of cash, one of the investors has also put in a certain amount of labour. In other cases, a certain amount of money is invested for a period of time and then a part of it is withdrawn again, etc. (see, for example, Swetz, 1989, p. 138 ff.). However, the axis corresponding to the formulation of the problems, and that corresponding to the methodology of resolution are not organized according to a more ample structure. The texts written by the abacus masters are essentially collections of problems. Interestingly enough, these problems are not framed by general theoretical principles. This is not only characteristic of abacist texts, but is also part of a way of thinking where reason and particular problems go hand in hand. Treatises on perspective are written along the same lines: they go from one problem to the next, trying to show the wide range of methods that can be applied *depending on the given examples* (see Elkins, 1994, p. 123).

Of course, this remark does not mean that abacist research was chaotic. The epistemological question is indeed to understand the organizing rationality behind problems and methods. In other words, the central question is: what does reasoning and knowing mean for an abacist? What kind of relationship does knowing have to the particular problems of abacist mathematical activity? Abacist pedagogy sheds light on these questions. In fact, this pedagogy was built on one principle intimating a certain conception of knowledge, as well as on another principle which demonstrates how knowledge can be acquired. These two principles, the one epistemological in nature, and the other methodological, are often echoed in an expression of the time referring to instruction "by reasoning and by example" (Bec, 1967, p. 290). This does not mean that there are no definitions at all that look for a kind of conceptual organization. Rather, this shows how definitions are quickly trapped in the web of examples. For instance, in the most developed texts, we find definitions of a few mathematical expressions. Piero della Francesca begins his *Trattato d'abaco* by demonstrating, by example, how to make calculations with fractions. He thereby explains expressions such as *schisare i rocti*, or to simplify fractions, and *ridurre a una natura*, or to reduce to a common denominator. After a problem has been formulated, he often begins by: *Fa' così* (do it like this), or *Volse fare così* (one wanted to do like this).

Consequently, we see that for the abacist, to reason does not mean to go syllogistically from proposition to proposition, using well-established truths as a starting point. Reason does not follow logically from a set of primary principles, as it did for the Greek geometers. On the contrary, once a problem has been outlined, reason manifests itself through the execution of the method that solves the problem. Mastering reason consists in being able to pose problems and to appropriately recognize the methods to solve them. Reason renders the world of individuals intelligible, not through abstract premises, but by tackling a diversity of problems as these problems arise and in the individuals' changing universe. Reason might show how to proceed *at the present moment* (as in problems on trade), or show how to divide the profits of an investment made by an association (each member having contributed a different amount of capital in the *past*), or even how to determine *future* interest payments on a loan. Hence, regardless of the modality of the action (i.e. regardless that the action relates to present, past or future events), the abacist mode of reason makes their universe intelligible, allowing the actions of individuals to be analyzed in terms of mathematical procedures. This is why abacus problems, as reflections of every day life, need to be understood in the context of an ontology and its ensuing cultural logic of meaning in which the concrete world is seen as dependant upon numbers. Enlightening in this respect is the fact that abacus problems are often designated by the term *ragione* whose general meaning during the Renaissance was

> . . .reason as the capacity to invoke the past (*narratio rerum gestarum*), to penetrate the present (*ragionare*) and thereby to foresee future events (*tener ragionamento*). This ratio implies a *ragionevole* order of the universe, which does not however exclude the part played by chance. [Reason] endeavours to distinguish between being and seeming, to know the nature of things in order to dominate and to exploit them despite venture and fortune. Well-suited to men who are *ragionevoli e intendenti*, it attempts, at its highest level, to penetrate *la ragione delle cose*. (Bec, 1967, pp. 324–325)

Thus, we see how, according to the mode of reason that underlies abacist mathematical problems, it is a matter of *demonstrating* how the numerical relations presented in the statement of the problem have to arrange themselves according to the method that governs them and reveal their eventual result. The abacists' mode of reason is not given as a way of theorizing the problems, or 'cases' or equations to which they 'belong,' but rather as a way of envisaging how one can resolve problems of diverse natures. It is through the *resemblances* between the situations described in the problems, that one will learn how to resolve them. The abacist episteme appears to be circumscribed by a way of thinking of the world in terms of resemblances and affinities, as described and discussed in depth by Foucault in *Les mots et les choses* (Foucault, 1966).

What about *generality* for the abacists? Mathematical generality, as seen by the abacist, can only be understood within the boundaries established by the mode of reason at work in a multitude of specific problems. It is not a kind of generality that organizes the world 'vertically', so to speak. Abacist generality appears as that which specifies the particular cases that conform to a rule. It is a generality that is arranged 'horizontally', expanding as the ever-increasing technical difficulties of the axis corresponding to methodology are overcome. The general and the particular become linked in knots that are formed in those places where there is an intersection of the axis of the formulation of problems and that of the methods used to solve them. This, however, is dependant on the space opened by reason, which, as we saw, is the reason of usages. It is the practical evidence of the adequacy between the particular and the general that plays the role of theoretical foundation. The abacists are still far removed from the world of Galileo, which can be understood *by* and *through* the language of mathem-

Figure 1. Illustration accompanying a problem in the *Trattato d'aritmetica* of Paolo dell'Abacco (14th century). Two labourers (on the left) are eating next to a fountain. One of the labourers has five loaves of bread and the other has four. A merchant joins their party. After having eaten three loaves together, the merchant departs and leaves five soldi for the labourers. "I wonder," says Paolo dell'Abacco "how much (of the money) each will get?" (Arrighi (ed.), 1964a, p. 82). This problem is a variant on the problems of companies.

atics. Mathematical language needed to become even more specific, and human production had to become even more automated before one could imagine the mechanist universe of the 16th century. Despite this, the abacists helped to formulate a vision of the world where human destiny could be modified (albeit only in relative proportions) by man's own actions, in stark contrast to the medieval standpoint which sat in contemplation of a transitory world governed by a supernatural order.

4. Abacist Semiotics

What can we say about signs, objects, and their relationship in abacist mathematics? In the preceding sections we have seen that to go from selective exchanges made on the spot, to rationally calculated trade required a specification of the usage value for merchandise and an elaboration of the abstraction of the concept of number itself. It is clear that the ideality of numbers during the Renaissance (and this is true for other mathematical objects as well) expresses an intellectual reflection of the world according to the forms of the activities that individuals pursue. However, even if the abacists' mathematical objects are, above-all, reflections of everyday life, the continuous variation in the formulation of problems led to the depiction of situations, which despite their realistic context, were almost impossible to find in the actual historical milieu of small-scale trade and commercial companies (see Figure 1). There is, in the continuous variation of problems, an aesthetic pleasure, quite evident in the situation described in Figure 1, which makes individuals' reflection of their world go beyond concrete situations. Although this aesthetic dimension is important to understanding thinking as a cognitive reflexive praxis (see Furinghetti and Radford, 2002) as well as what we mean by an intellectual reflection of the world, I will not dwell on this aspect of abacist semiotics. Instead, let me remark that, as a result of this reflexive praxis, for the abacist, numbers are no longer thought of as Greek monads, but become divisible. Thus, the ducati used to solve the problem of Figure 1 are fractions of the soldi. A break occurred with the Greek way of thinking that the Renaissance humanists had been actively pursuing and rediscovering through the manuscripts they acquired from the lands of the eastern Mediterranean[12]. It was this break that allowed fractions to enter the mathematical universe. While Renaissance mathematicians never lost their high esteem for the Greek geometers, and even gave definitions of numbers 'in the Greek fashion'[13], in practice, they considered fractions true mathematical objects[14]. We can argue that, in abacist practice, instead of referring to pre-existing objects (to *eidos*), mathematical objects become materially produced by the actions of individuals. While the object of the sign in Greek semiotics belongs to the world of *eidos*, that is, to a world of objects thought of as deprived of all sensual origin, for the abacist, in contrast, the objects that he designates with words from natural language or with signs from Indo-Arabic arithmetic belong to the concrete world, even in the case of non-applied problems, or ones that have no relation to commercial content or to the content of exchange. Decidedly, the abacist's mathematical object is a worldly conceptual object.

To help our discussion of the abacist relationship between object and sign, let us go back to the trade problem seen earlier in Piero della Francesca's *Trattato d'abaco* and take a detailed look at its solution. To review, this problem involved an exchange where one of the participants was offering wax with a value of 9 1/4 ducats and a barter value of 10 2/3. The other participant offered wool with a barter value of 34 ducats. The goal was to determine its value in cash, in the context of fair trade. To solve the problem, della Francesca says:

Say this. The hundred of wax is worth 9 1/4 and it was put at 10 2/3, therefore: 10 2/3 gives 9 1/4, what would give 34? Multiply 34 by 9 1/4 that makes 3774 twelfths; make twelfths of 10 2/3 and it's 128, divide 3774 by 128 and you get 30 47/64 (*sic!*); that's how much the thousand of wool is worth cash: 30 47/64 (*sic!*) ducati. I made twelfths because there are thirds and quarters.

(Arrighi (ed.), 1970, p. 51)

The sign (for example, the word of written language and the arithmetical symbol) seems to be a transcription of oral production. It refers to actions that someone has to carry out on certain mathematical objects. This is underlined by expressions such as 'multiply' and 'do,' etc. Up to now, none of this is unusual. For example, if we consider Babylonian mathematical texts, we also find instructions pointing to specific actions that someone has to carry out on certain objects. However, the difference lies in the fact that Babylonian arithmetic takes its meaning from the representation of figures (e.g. rectangles, squares) that are cut up and displaced. As Høyrup (1990) has shown, there is, from the beginning, a *visual* referent (even if it is not included in the written text). The object is represented iconically (Radford, 2001). In this way, the object is presented to our understanding; it becomes 'sensible' (Kant, 1781, 1787/1996) in an act of representation that allows us to establish a link of deep signification between the intended object and its representation (Husserl, 1900/1970). By virtue of its iconicity, the object is endowed with a meaning that authorizes certain ways of treating or processing (as understood by Duval, 1993) representations. In such cases, representations impose certain visual facets of the object on psychological activity, so that the referent does not become lost. Within the mass of abacist problems that are similar to the one that we are analyzing, the referent, in contrast, has to be voluntarily suppressed. The multiplication of 34 and 9 1/4, for example, has no concrete meaning. In order to complete this mathematical operation, one has to consent to losing the referent. Indeed, what can 34 ducats times 9 1/4 ducats actually give?

The second important point is that mathematical objects take on a kind of autonomy. They are treated as entities that produce things in themselves: "10 2/3 *dà* 9 1/4, *che darà 34*?" The verb 'to give' (*darà, dare*) is central here. A number *gives* a certain number. Therefore, how much will a third number *give*?

Where does this creative power of numbers come from? Anthropologists have often underlined the fact that in certain 'animist' cultures, objects are seen as having a life of their own. It would certainly be an exaggeration to say that this is also the case when it comes to the abacist concept of numbers. It would be more correct to view the momentum of numbers within the context of emerging capitalism and its consequent division of labour, where money *makes* money. In fact, in the conceptual structures that follow the division of labour, the subject sees itself distanced from the product of human activity. A lengthy chain of human activity interposes itself between subjects and objects. The relation between subject/object thus becomes strictly mediatized by a social organization, which, in the case we are considering, makes the value represented by the numeric signs (10 2/3, 9 1/4 ducats, etc.) appear to be an entity that reproduces itself on its own[15].

To better grasp the semiotic phenomenon that we have been describing, we can contrast it with the conceptualization of multiplication in other cultures. The case of Babylonian mathematics is especially enlightening. One of the conceptualizations of the multiplication is rendered by the standard expression ‘x a-rá y’ where the expression ‘rá’ corresponds to the verb ‘to go.’ Høyrup notices that in several Seleucid texts, the meaning of multiplication is determined in relation to the number of x steps each step having a length equal to y. Thus, ‘x a-rá y’ corresponds to ‘x steps of y.’ In a second conceptualization, multiplication takes on the form of duplication or repetition. In yet a third conceptualization, the idea behind multiplication (for example, multiplication by a constant or by the reciprocal of a number) is that of raising or lifting. Høyrup remarks that

> The connection between ‘raising’ and multiplication is not obvious to the modern mind [. . .] One clue derives from the way volumes are calculated. If the base is quadratic, rectangular or circular, [during the calculation of the volume] it is normally [seen as] ‘spanned’ by length and width. (Høyrup, 1990, p. 47)

From a semiotic viewpoint, the preceding considerations suggest that the relationship between object and sign does not find its source exclusively in language and in the objectifying word. Within the social discourse where they are lodged, words exceed empirical reality, and provide operations between numbers with a certain autonomy, thereby creating another reality. This creative effect on the part of objectifying words and mathematical signs is actually a consequence of the abstraction that results from the system of human activities (a system which includes the aesthetics of semiotic representation and the contextual logic of cultural meaning).

When discussing objects in general, Leont’ev says:

> Objects themselves can become stimuli, goals, or tools only in a system of human activity; deprived of connections within this system they lose their existence as stimuli, goals, or tools. (Leont’ev, 1978, p. 67)

While for the Babylonians, multiplication becomes enveloped in metaphors issuing from concrete and sensuous actions, such as walking, or in the case of calculating volume, digging or even shifting sand, for the abacists, multiplication is linked to a conceptualization of numbers as they relate to the profoundly mediatized practices of emerging capitalism.

We should try to explain the preceding remarks in more general terms. The conceptualization of the object of the sign and its manner of denotation appear to be linked to social practice. The ‘acts which impart sense’ to take up an expression used by Husserl (1900/1970) are located within the cultural categories that individuals form through their activities. In the case of the abacists, the object of the sign and its mode of designation are directly related to the concept of value that is engendered by the new relations produced by the division of labour, and the central role taken up by trade and production. An important consequence of the changes introduced by abacists, which are both conceptual (that is, with regards to the *object* of the sign) and semiotic (that is, with regards to modes of *designation*) is that there is a new mathematical treatment of non-homogenous quantities. In the next section, we will see that this was made possible precisely because of a homogenization of the world that resulted from the concept of value.

5. The Loss of Homogeneity

No reader of Euclid has failed to notice the complex definition that he gives of proportional magnitudes. Among other things, Euclid insists that the ratios be of the same type. Hence, one is not allowed to compare objects of differing quality. Since Aristotle, the Greek episteme ordered the world according to a system of categories that divided it into compartments. The abacists were not to respect this principle, even though they declared themselves to be the faithful successors to the Ancients. Hadden remarks that

> Niccolo Tartaglia (d. 1557), for example, formulates a statics problem in which it is required to calculate the weight of a body, suspended from the end of a beam, needed to keep the beam horizontal. Tartaglia’s solution requires the multiplication and division of feet and pounds in the same expression. Euclidean propositions are employed in the technique of solution, but Euclidean principles are also thereby violated. (Hadden, 1994, p. 64)

What renders the multiplication of distances by weights possible? As a means of explaining this conundrum, Hadden shows that, during the Renaissance, there was no longer any reason to uphold the separation between objects of differing natures. Behind this conceptual change was the idea of an ‘equalization’ of objects engendered by money. He further explains by arguing that money was conceptualized differently in Renaissance social practice, than it was in classical Greece. Relying on Aristotle’s reflections in his *Nicomachean Ethics*, he points out that when Aristotle discusses problems of corrective justice issuing from certain kinds of private exchange (such as transactions for buying and selling), where a judge might be obliged to determine the penalty for someone who had behaved wrongfully during a transaction, the ‘just’ position is formulated using reason, modeled according to examples in the calculations of proportions, such as an intermediate point between gains and losses[16]. This treatment of justice distinguishes itself from distributive justice, as discussed by Aristotle in Book V, 6 of the *Nicomachean Ethics*, where, as the philosopher says, one has to consider “the value of people themselves” (e.g. their rank) when looking for the just position. Hadden adds:

> Labour is seen here as social but not as equal in determining value; in fact, the labour *per se* does not appear as a concept in Aristotle’s *Nichomachean Ethics*, but the significant term in the equation is the status of the producer relative to other producers. (Hadden, 1994, p. 78)

Money then becomes a measure for making goods commensurable. However, its nature remains defined by *pure human convention*:

> Money, then, acting as a measure, makes goods commensurate and equates them; for neither would there have been association if there were not exchange, nor exchange if there were not equality, nor equality if there were not commensurability. Now in truth it is impossible that things differing so much should become commensurate, but with reference to demand they may become so sufficiently. There must, then, be a unit, and that fixed by agreement (for which reason it is called money); for it is this that makes all things commensurate, since all things are measured by money.
> (Aristotle, *Nicomachean Ethics*, 1133^b, 16–23, tr. Ross, 1947)

It is precisely this conventional character that Aristotle gives to money, that makes it impossible to calculate proportion in problems about nature. In his *Physics*, the comparison of quantities that differ categorically is forbidden. Nature becomes partitioned into conceptual categories of genus and species.

When contrasting the conceptualization of money in classical Greece with its conceptualization during the Renaissance, Hadden emphasizes that in the latter period, money is seen as being part of the nature of things. He makes reference to the work *De Moneta* by the prelate Nicole Oresme, where the author vigorously affirms that rather than a nominal value, money has a value which is *real*[17]. As a result, objects can be compared, even if they are different in *appearance*. Weight and distance, for example, can be multiplied. Within the abstraction that is subsequently engendered, the time devoted to labour by individuals becomes merchandise, and its value can be added to other values.

There is a revealing problem in Piero della Francesca’s *Trattato d’abaco* that states the following:

> A gentleman hires a servant on salary, that he is to pay 25 ducati and one horse per year; after 2 months the worker says that he does not want to remain with him

anymore and wants to be paid for the time he did serve. The gentleman gives him the horse and says: give me 4 ducati and you shall be paid. I ask, what was the horse worth? (Arrighi (ed.), 1970, p. 107)

Here we clearly see how time spent working can be considered in terms of merchandise. It can be added to and exchanged and becomes as concrete as any other earthly object[18].

As was the case with Aristotle, money comes to measure the objects that fill the universe of individuals. However, during the Renaissance, it is no longer simply a convention: money, as a semiotic signifier of the concept of value, belongs to the class of things coming from nature and from the work of individuals. Thereby, it is possible to conceive of nature as being, in a sense, homogenous[19].

However, what makes this transformation possible? What are the conditions of possibility that make one think of nature as a homogenous entity?

After a long discussion of the role played by money in measuring the value of merchandise during the Renaissance, Hadden (1994, p. 65) suggests that the transformation that forms the basis for the homogenization of the world – a homogenization that empowers the calculation of proportions regardless of the nature of the objects under consideration – can be found in the new structures of social relationships established by individuals. More specifically, it is the relationships governing the production of merchandise, exchanges of merchandise, as well as the new practices regulating transactions that are responsible for the profound epistemological changes at the basis of the transformation in the way of thinking about the world. Nevertheless, I wish to suggest that these conditions of possibility should not be understood as the 'last logical conditions.' For, far from being beyond historicity, these conditions are formed during their own historical moment in a particular dialectical process in which the logic of production and material conditions merge with the logic of cultural significations – that is, with a logic of meaning[20].

6. Synthesis and Final Remarks

This article set itself the epistemological task of exploring the relationship between language, knowledge and social practice. This exploration was motivated in part by one of the fundamental propositions made by certain currents issuing from 'the paradigm of language.' This proposition holds that it is the discursive activity, through the production of symbols, which creates mathematical objects. The aim was to find out whether or not such a power of creation could be attributed to discursive activity. In the first section, I presented an objection to this theoretical position[21]. Even in its most nuanced form, which adds a kind of back and forth movement between the objects thus created and the symbols produced by discourse, the proposition necessitates that we take other elements into consideration – elements found *beyond* discourse itself. We were urged on by Leont'ev's remark reminding us that behind speech acts, we find social practice.

We have anchored our objection to the realization that the way in which individuals reflect on their world – a reflection that is at the basis of the production of knowledge – is not objectified by words and symbols alone. This act of reflection enters into the system of social knowledge through various avenues, and not exclusively through verbal expression and writing. As Ilyenkov (1977) observes, knowledge is also objectified by other means, such as sculptural forms, graphics and plastics, as well as by the habitual and historically constituted manner in which one acts towards things and interacts with individuals. To this end, such a reflection "is expressed not only in words, in speech and language, but also in drawings, models and (other) symbolic objects" (Ilyenkov, 1977, p. 79).

This remark does not mean that language does not possess any kind of creative power; rather, it underlines two important points. *Primo*, it shows that language is *one* of the means of objectification (albeit a very important one), but that there are also several others[22]. *Secundo*, we see, that as a means of objectification, language does not objectify indiscriminately. Language, like any other semiotic system, functions inside a cultural network of significations, from whence grammar and syntax draw their meaning[23].

To elaborate our argument, it was necessary to illustrate the objectifying movement of language in its historical-cultural context. Keeping in mind our epistemological objectives, we were compelled to outline the possibilities of language by associating them to social practice. Perhaps a historical approach is not indispensable for this kind of endeavor. Surely there are other means of understanding language in its original way of operating. To this end, Merleau-Ponty suggests an interesting course when he says, "if we want to understand language in its original way of operating, we must pretend never to have spoken,... [we must] compare the art of language to the other arts of expression, and try to see it as one of these mute arts." (Merleau-Ponty, 1960, pp. 58–59). Thanks to the feeling of otherness it brings, history provides us with just such an opportunity for comparison. We chose the case of mathematics during the Renaissance, and to grasp the role of language in abacist thought, we began by examining the social relations and production of the period[24]. In this context, we sought to highlight certain characteristic traits of the abacist 'mode of reasoning' and of its corresponding semiotic space. Our search for the conceptual object thematized by abacist words and discourse was correlated to the ideal phenomena that social practice was seeking to consolidate. In the course of our analysis, we saw how the concept of value was central to abacist thought. The concept of value belongs to the practical knowledge characteristic of that period in history, preoccupied, in its most speculative aspects, with the restoration and interpretation of ancient thought. Renaissance humanism is situated between two poles: one which looks back admiringly at the accomplishments of the Ancients, and another, which looks forward and tries to move beyond these venerated achievements[25]. It is situated within this *will to truth* (Foucault, 1971) that is carried along by cultural institutions – not only in the courts and libraries of princes, but also in the small-scale trade and sale marketplaces of merchants – through which the individual establishes a new relationship with his world. It is a relationship which situates knowledge (*savoir*) both as a form of action and as a personal accomplishment[26].

At the centre of the *will to truth*, between the spheres of action and of personal life, where modernity's idea of subjectivity began to be formulated in the Renaissance, we encounter the concept of value. This concept came to establish a new form of *resemblance* between things. This resemblance introduced categories other than those of *convenientia*, *aemulatio*, *analogie* and *sympathie*, which Foucault (1966) deemed characteristic of the pre-modern period. Wax and wool are different; they have a different usage value. However, it is possible to see them as the *same*, and herein we note the role of value. This tremendous abstraction found its historical conditions of possibility in the social organization of forces of production during the Renaissance, or more specifically, in the forms of functioning of methods of production – although without being reduced to them. From that moment on, signs and their objects participated in a previously unthinkable relationship.

The appearance of a new type of resemblance supplied by the concept of value (a resemblance that we could call *formal*) leads to a space of historical possibilities from whence a symbolic algebraic language emerges (a language to carry out calculations based on an arbitrary type of designation without any resemblance to the designated object). This formal resemblance, I wish to suggest, is part of a semiotic phenomenon that begins to surface towards the end of the Trecento with the development of commerce and exchange. This makes it possible for one to speak of the arbitrary nature of money, as Oresme does, and to imagine a new representation of space (such as manifests itself in Giotto's work, for example: see Radford, 1997). Although the abacist does not map out a symbolic algebraic language, he opens up a space were the arbitrary symbols of Bombelli, Viète, Stifel and Descartes can situate themselves. In the abacist use of the word 'thing' there is no more resemblance between signifier and signified. Of course, the word 'thing' has its source in previous mathematical practices (Gandz, 1926, 1928). Abacists did not deny the Arab influence on their work. Nevertheless, the abacist signified moves itself in a discursive space endowed with new dimensions made possible by the new forms of social organization of the forces of production and a concomitant logic of significations. This new signified is what we have called the object of the sign and is that which we have tried to outline during our discussion of the abacist 'mode of reason' and its semiotics.

The new category of representation that infiltrates abacist semiotics during the Trecento makes it possible to go beyond the iconicity of medieval semiotics. Hardly the only participant in this transformation, the abacist nevertheless makes a decisive contribution to preparing a terrain where, as Foucault says, words and things will separate. However, the preparation of this fissure is not just the result of a language that has begun to dislocate itself. Between language and that which it thematizes – that is to say, between sign and object – lies social practice.

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NOTES

1. Of course, the advent of the paradigm of language does not mean that questions about the relationship between cognition and language have never been considered before. For instance, contemporary discussions about the cultural and biological origin of language and about the role of language in cognition go back to the seventeenth century. It is during this period that Leibniz and Vico rejected the 'arbitrary' conception of language, that is to say, the conception which underlined the key role played by the social and cultural dimensions conveyed by language in our experience of the world. Rather, Leibniz and Vico advocated the 'naturalist' position, which instead stressed that linguistic activity is rooted in our biological make-up, and already operates on a level that takes precedence over any cultural experience (see Gensini, 1995, p. 7).
2. Although with often notable differences, this is the case when it comes to certain interactionist and discursive approaches (see e.g., Gilly et al., 1999). We should note that the idea that conceptual objects are inseparable from language is also one of the cornerstones of postmodernism. According to this perspective, "social life derives its meaning from speech acts. Rather than a conduit, language is a creative force" (Murphy, 1988, p. 603).

3. Such conceptualizations encompass positions as disparate as the anti-psychologism of Wittgenstein (1967) and Frege (1971) or the psychologism of the young Husserl at the time of *The Philosophy of Arithmetic* (Husserl, 1891/1972).
4. "In the fourteenth century," says Adrian Room in the *Dictionary of Changes in Meaning*, "a 'counter,' as the name suggests, was an object used in counting or in keeping accounts, i.e. like the 'counters' in some modern games or in marking cards. From this, the sense developed naturally to mean first, a desk where money was counted, then a money-changer's table, and finally a tradesman's or shopkeeper's 'surface' where goods were served. The current sense arose from the fourteenth century" (Room, 1986, pp. 72–3). I am grateful to Louis Charbonneau for bringing the etymology of the word "counter" to my attention.
5. This is the case for research stemming from 'situated cognition'. For some paradigmatic works see Brown et al. (1989), Hutchins (1995), Lave (1988) and Lave and Wenger (1991). For some works on 'distributed cognition' see Salomon (1993) or even others such as those of Cole (1996) and Engelström et al. (1999) inspired by Leont'ev (1978) and his theory of activity.
6. See, for example, Otte (1994), Lizcano (1993), Høyrup (1995), Szabó (1977).
7. Restivo provided an analysis of certain mathematical practices (1992, 1993, 1998). Arguing from a sociological perspective and inspired by dialectical materialism, discussed in detail the role of power and class struggle in the use that certain societies make of mathematics. There is no doubt that Restivo's argument has many interesting facets. However, our theoretical framework and focus on the epistemological problems of research are different in this article.
8. For a discussion of Plato's familiarity with the Orient in general, see Whitney (1979). There is a discussion of the oriental heritage in Greek thought in Bernal's work (1987). See in particular volume 2, pp. 106 ff. With regards to mathematics, see Katz (1996).
9. The economist Nicholas Barbon said, in his treatise 'A Discourse of Trade,' published in 1690, that the value of all merchandise comes from its use.
10. Pacioli and Cardan constitute two rare exceptions among those abacists who managed to carve a place for themselves in the higher spheres of the society of the period. Towards the end of his life, Cardan nevertheless preferred to give up his career as a mathematician and devoted himself to practicing medicine (Fierz, 1983), which was considered a much more socially acceptable endeavor than mathematics. Participation in the higher social spheres was more difficult for abacists, since the courts of Italian cities preferred to surround themselves with those mathematicians who were translating Greek manuscripts sought out in the Levant.
11. In this way, an abacist could himself become an entrepreneur (see Goldthwaithe, 1972–73).
12. During the hunt for Greek manuscripts, the sovereigns of Italian cities often sent emissaries to find them. Others went without patronage. In this context, Florence quickly became the most important market for classical manuscripts. For example, we know that during his second voyage in the East (1421–23), the Sicilian Giovanni Aurispa brought back 238 Greek manuscripts, including a codex of the Iliad and a manuscript of the *Mathematical Collection* of Pappus (Rose, 1975, p. 28). Filefo, an attaché of the Venetian embassy in Constantinople in 1420–27, returned with a stack of manuscripts by more than 40 Greek authors (Pfeiffer, 1976, p. 48). These manuscripts were often kept in private libraries. One of the most complete libraries of the period was that of Duke Federico da Montefeltro d'Urbino (1422–1482). It was in this library that Luca

Pacioli, who had been called to court, was able to consult a translation of Archimedes and the algebra of al-Khwarizmi.

13. In a 12th century manuscript edited by Arrighi (1964b), the author, maestro Gugliemo defines numbers as follows: "Numerus est unitatum collective vel quantitatis acervus ex unitatibus profusus."
14. This does make for a certain incoherence, but does not go any further. Thus, in a passage by Fibonacci, where he explains what a number is, he begins by giving the traditional Greek definition in terms of multiple units, and consequently finds it difficult to apply this concept to money problems, as these problems often involve separate parts of a given unit, as was the case with the monetary system of the times (see Arrighi, 1965, p. 381).
15. Donham (1999) has demonstrated the precise character of the act of multiplication of money as it issues from the abstraction brought on by the process of dividing labour. He notes that capital *qua* capital does not produce anything, since it is the product of other workers. However, the discourse surrounding the value of merchandise makes for the fact that "capital seems to acquire a life of its own" (Donham, 1999, p. 103). There is perhaps no better example of the kind of discourse that proposes the generative and multiplicative character of money, than that which merchants used when referring to interests rates. How was one to fix such rates? Hadden mentions the thirteenth century work *De contractibus usurariis* by Olivi, where the author contends that the profits from a loan are legitimate if the same profits could have been obtained by investing the money. In this case, money seems to produce money (Hadden, 1994, pp. 99–100). A very interesting discussion of the semiotics of paper money can be found in Rotman (1987). Nevertheless, these remarks neither amount to saying that capitalism is the logical sufficient condition for the emergence of abacist mathematical thinking, nor that the emergence of abacist mathematical thinking can be stated in terms of the Marxist relationship between infrastructure and superstructure, simply because the cultural conditions allowing an episteme to emerge do not follow the logic of causal relations. In all likelihood, as suggested by Lizcano, the problem of the emergence of a mode of thinking cannot be reduced to the problem of its "last explicative motive" (Lizcano, personal communication, October 15, 2002). I will come back to this point later.
16. "Now the judge restores equality; it is as though there were a line divided into unequal parts, and he took away that by which the greater segment exceeds the half, and added it to the smaller segment. And when the whole has been equally divided, then they say they have 'their own'–i.e. when they have got what is equal. The equal is intermediate between the greater and the lesser line according to arithmetical proportion." Aristotle, *Nicomachean Ethics*, 1132^{a} 23–29 (cited here from the translation of W.D. Ross, 1947).
17. Hadden cites the following passage from *De Moneta* (translated into English from the Latin by Charles Johnson): "For if a man gives bread or bodily labour in exchange for money, the money he receives is as much his as the bread or bodily labour which he (unless he were a slave) was free to dispose. For it was not to princes alone that God gave freedom to possess property, but to our first parents and to all their offspring, as it is in Genesis." (Oresme, 1956, pp. 10–11)
18. Here is della Francesca's solution. (The little dash that he places on top of certain numbers is used to represent unknown quantities). Do this. You know that he has to give him 25 ducati per year, for 2 months it comes to 4 I/6; and the horse put that it's worth $\bar{\text{I}}$ thing, for 2 months it is worth 2/12 of the thing that is I/6. You know that you

have to have in 2 months 4 ducati and I/6 and I/6 of the thing [that is 4 I/6 ducati and I/6 of the thing]. And the gentleman wants 4 ducati that added to 4 I/6 makes 8 I/6, because you have I/6 of the thing and in order to get Ī there are 5/6; therefore 5/6 of the thing is equal to 8 I/6 number. Reduce to one nature [i.e. to a whole number], you will have 5 things equal to 49; divide by the things it comes out to 9 4/5: the thing is worth so much and we put that the horse is worth Ī, therefore it is worth 9 ducati 4/5 of a ducato." (Arrighi (ed.), 1970, p. 107)

19. To show the distinctive and local character resulting from the homogenization of products during the Renaissance, Heilbroner mentions the case of the Maoris of New Zealand, whom he says, "you cannot ask how much food a bonito hook is worth, for such a trade is never made and the question would be regarded as ridiculous." (Heilbroner, 1953/1999, p. 27)
20. As stated in endnote 16, a mode of thinking cannot be reduced to the material conditions of the culture. Although signs are produced within the limits and possibilities of the "technology of semiotic activity" (Radford, 2002), they convey meanings that are entangled in the symbolic component of the culture and its logic of meaning as it is subsumed in what I call *Cultural Semiotic Systems* (see Radford, 2003a).
21. We easily see that in its most radical form, this position often reduces reality to one which is constructed by words. The analysis of the construction of knowledge then becomes centered only on 'immediate interaction' and on discursive analysis. This position leads to some important theoretical problems. For instance, I cannot see how this position can avoid reducing epistemology to a kind of verbal behaviorism and reality to an arbitrary nominalism (see also Radford, 2000a).
22. The signs relating to other means of objectification are naturally articulated differently from signs relating to language. The relation to time and space fostered by signs relating to the figural register, for example, differ from language's relation to time. In a discussion of Piero della Francesca's famous painting *Virgin in Mantel*, Francastel notes that time provides language with a kind of continuum to organize itself. Words are uttered one after the other. In painting the situation is completely different. There is no equivalent continuum to support the entire figurative system. The montage, that is the global organization of the different parts of the painting, becomes the signifying tool. "From whence," Francastel concludes, "it results that we could no longer consider that language constitutes the typical form of human thought." (Francastel, 1967, pp. 159–160)
23. For a detailed elaboration of the view of language as *one* means of objectification with modifications engendered by the two points of our argument, see Radford (2000b, 2002 and 2003b).
24. "Production is the privileged point of entry for the comprehension of social totalities, for the understanding of how people make their own history, even if it is not as they would have chosen it." (Donham, 1999, p. 58)
25. "In the Renaissance (...) one finds an intensification of antiquity's influence, that which brings to a new affirmation man's dignity, by means of the consideration of his superiority over all of nature. Such superiority consists in the creative capacity of man, that which determines the formation of culture and manifests itself in an infinite progress. In this [creative capacity] consists for the Renaissance writers man's excellence. An excellence that presents itself bound to human activity in the conquest of knowledge and learning in order to affirm itself as the ideal of man's spiritual elevation." (Mondolfo, 1963, pp. 237–238)

26. Thus, addressing himself to his son in a passage of his memoirs, the tradesman Giovanni di Paolo Morelli says: “Once you have left school, make sure, that for at least one hour each day, you study Virgil, Boccaccio, and Seneca, or other authors as [you did] at school. This will lead to a great development of your mind (. . .) once you have attained maturity and your intelligence has begun to taste the reason of things and the sweetness of science, you will experience as much pleasure, joy and comfort as you get from all that you possess.” (G. Morelli, cited by Bec 1967, p. 293)

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